steepestdescent

June 16, 2020

1 Steepest descent

Consider $f \in C^1$. The steepest descent consists to iteratively compute

$$x_{k+1} = x_k - \alpha^* \nabla f(x^k)$$

where $\alpha^* \in \arg\min_{\alpha \geq 0} f(x_k - \alpha \nabla f(x_k))$.

[20]:
```
using Optim
using Plots
plotly()
#pyplot()
```

[20]:
```
Plots.PlotlyBackend()
```

We need the LinearAlgebra library to access some methods as det, that computes a matrix determinant.

[21]:
```
using LinearAlgebra
```

1.1 Example 1

Consider the bivariate function

$$f(x, y) = 4x^2 - 4xy + 2y^2$$

[22]:
```
f1(x) = 4x[1]*(x[1]-x[2])+2*x[2]*x[2]

default(size=(600,600), fc=:heat)
x, y = -2.5:0.1:2.5, 0.5:0.1:2.5
z = Surface((x,y)->f1([x,y]), x, y)
surface(x,y,z)
```

Its gradient is

$$\nabla f(x, y) = \begin{pmatrix} 8x - 4y \\ 4y - 4x \end{pmatrix}$$

The Hessian is

$$\nabla f^2(x, y) = \begin{pmatrix} 8 & -4 \\ -4 & 4 \end{pmatrix}$$

[8]:
```
A = [8 -4; -4 4 ]
```

[8]:
```
2×2 Array{Int64,2}:
  8  -4
 -4   4
```

The principal minors determinants are

[9]:
```
8
det( A )
```

[9]:
```
16.0
```

Therefore, the matrix is positive definite. We can confirm this by computing the eigenvalues:

[10]:
```
eigvals(A)
```

[10]:
```
2-element Array{Float64,1}:
  1.5278640450004206
 10.47213595499958
```

We compute the gradient as

[11]:
```
function f1grad(x)
    return [8*x[1]-4*x[2], 4*x[2]-4*x[1]]
end
```

[11]:
```
f1grad (generic function with 1 method)
```

Consider $x_0 = (2, 3)$. Therefore $\nabla f(x_0) = (4, 4)$.

We have to minimize the univariate function

$$m(\alpha) = f((2, 3) - \alpha(4, 4)) = f(2 - 4\alpha, 3 - 4\alpha)$$

The derivative of $m(\alpha)$ is

$$\begin{aligned}
m'(\alpha) &= \nabla_{(x,y)} f(2 - 4\alpha, 3 - 4\alpha)^T \nabla_\alpha \begin{pmatrix} 2 - 4\alpha \\ 3 - 4\alpha \end{pmatrix} \\
&= \begin{pmatrix} 8(2 - 4\alpha) - 4(3 - 4\alpha) & 4(3 - 4\alpha) - 4(2 - 4\alpha) \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} \\
&= -\begin{pmatrix} 4 - 16\alpha & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} \\
&= -16 + 64\alpha - 16 \\
&= 64\alpha - 32
\end{aligned}$$

The second derivate of $m(\alpha)$ is

$$m''(\alpha) = 64$$

Therefore the unidimensionel model is strictly convex. The minimizer can be found by setting $m'(\alpha^*) = 0$, leading to $\alpha^* = \frac{1}{2}$. Therefore

$$x_1 = x_0 - \frac{1}{2}\nabla f(x_0) = (2,3) - \frac{1}{2}(4,4) = (0,1),$$

and

$$\nabla f(x_1) = \begin{pmatrix} -4 \\ 4 \end{pmatrix}$$

The univariate function to minimize is now

$$m(\alpha) = f((0,1) - \alpha(-4,4)) = f(4\alpha, 1-4\alpha)$$

and its derivative is

$$\begin{aligned} m'(\alpha) &= \nabla_{(x,y)} f(4\alpha, 1-4\alpha)^T \nabla_\alpha \begin{pmatrix} 4\alpha \\ 1-4\alpha \end{pmatrix} \\ &= (8 \times 4\alpha - 4(1-4\alpha), 4(1-4\alpha) - 4 \times (4\alpha)) \begin{pmatrix} 4 \\ -4 \end{pmatrix} \\ &= (-4 + 48\alpha, 4 - 32\alpha) \begin{pmatrix} 4 \\ -4 \end{pmatrix} \\ &= -32 + 320\alpha \end{aligned}$$

The root of $m'(\alpha)$ is $\alpha^* = \frac{1}{10}$, and $m''(\alpha) = 320$, thus α^* is a global minimizer. We obtain

$$x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{10}\begin{pmatrix} -4 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{4}{10} \\ \frac{6}{10} \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{3}{5} \end{pmatrix}$$

We could continue, but such a hand computation is tedious. We will automatize the procedure by constructing a Julia function.

[12]:
```
function steepestdescent(f::Function, fprime::Function, x0, h::Float64, verbose:
 ↪:Bool = true,
                         record::Bool = false, tol::Float64 = 1e-7, maxiter::
 ↪Int64 = 1000)

    function fsearch( ::Float64)
        return(f(x- *grad))
    end

    x = x0
    k = 0

    grad = fprime(x)

    if (verbose || record)
        fx = f(x)
    end
    if (verbose)
```

```
        println("$k. x = $x, f($x) = $fx")
    end
    if (record)
        iterates = [ fx x' ]
    end

    while ((k < maxiter) && (norm(grad) > tol))
         = Optim.minimizer(optimize(fsearch, 0, h, GoldenSection()))
        x = x- *grad
        k += 1
        grad = fprime(x)
        if (verbose || record)
            fx = f(x)
        end
        if (verbose)
            println("$k. x = $x, f($x) = $fx")
        end
        if (record)
            iterates = [ iterates; fx x' ]
        end
    end

    if (k == maxiter)
        println("WARNING: maximum number of iterations reached")
    end

    if (record)
        return x, iterates
    else
        return x
    end
end
```

[12]:
```
steepestdescent (generic function with 5 methods)
```

The following variant proposes to enlarge the interval where the unidimensional search is done when the upper bound is reached.

This is only valid for convex functions!

But the idea will be adapted and generalized when discussing about trust regions.

[13]:
```
function steepestdescent_convex(f::Function, fprime::Function, x0, h::Float64,␣
 ↪verbose::Bool = true,
        record::Bool = false, tol::Float64 = 1e-7, maxiter::Int64 = 1000)

    function fsearch( ::Float64)
        return(f(x- *grad))
```

```
    end

    x = x0
    k = 0

    grad = fprime(x)

    if (verbose || record)
        fx = f(x)
    end
    if (verbose)
        println("$k. x = $x, f($x) = $fx")
    end
    if (record)
        iterates = [ fx x' ]
    end

    Δ = 1e-6

    while ((k < maxiter) && (norm(grad) > tol))
          = Optim.minimizer(optimize(fsearch, 0, h, GoldenSection()))
        while ((h- ) <= Δ)
            h *= 2
              = Optim.minimizer(optimize(fsearch,  , h, GoldenSection()))
        end
        h = 
        x = x- *grad
        k += 1
        grad = fprime(x)
        if (verbose || record)
            fx = f(x)
        end
        if (verbose)
            println("$k. x = $x, f($x) = $fx")
        end
        if (record)
            iterates = [ iterates; fx x' ]
        end
    end

    if (k == maxiter)
        println("WARNING: maximum number of iterations reached")
    end

    if (record)
        return x, iterates
    else
```

```
        return x
    end
end
```

[13]: `steepestdescent_convex (generic function with 5 methods)`

Executing this function on the problem, we obtain

[14]:
```
sol, iter = steepestdescent(f1, f1grad, [2.0,3.0], 2.0, true, true)
```

```
0. x = [2.0, 3.0], f([2.0, 3.0]) = 10.0
1. x = [3.3384826014781765e-9, 1.0000000033384826], f([3.3384826014781765e-9,
1.0000000033384826]) = 2.0
2. x = [0.4000000056568157, 0.5999999996847565], f([0.4000000056568157,
0.5999999996847565]) = 0.4000000042732578
3. x = [1.2686234840408872e-8, 0.20000004191604126], f([1.2686234840408872e-8,
0.20000004191604126]) = 0.08000002338384717
4. x = [0.08000002003440877, 0.12000002949337335], f([0.08000002003440877,
0.12000002949337335]) = 0.01600000792444612
5. x = [1.2797865406311004e-9, 0.04000001297171028], f([1.2797865406311004e-9,
0.04000001297171028]) = 0.003200001870708075
6. x = [0.016000005590790508, 0.024000008148791697], f([0.016000005590790508,
0.024000008148791697]) = 0.0006400004396667061
7. x = [4.5701057174718507e-10, 0.008000003964587964],
f([4.5701057174718507e-10, 0.008000003964587964]) = 0.00012800011224250156
8. x = [0.0032000017064671205, 0.004800002532327185], f([0.0032000017064671205,
0.004800002532327185]) = 2.560002712829075e-5
9. x = [4.110368252360774e-11, 0.0016000009764577262], f([4.110368252360774e-11,
0.0016000009764577262]) = 5.120005986267632e-6
10. x = [0.0006400004046497921, 0.0009600005964701434],
f([0.0006400004046497921, 0.0009600005964701434]) = 1.0240012814339186e-6
11. x = [1.6262869089594956e-11, 0.00032000025010139714],
f([1.6262869089594956e-11, 0.00032000025010139714]) = 2.048002993134258e-7
12. x = [0.00012800010446232586, 0.0001920001553967927],
f([0.00012800010446232586, 0.0001920001553967927]) = 4.0960066523961365e-8
13. x = [1.2406358492617064e-12, 6.40000573618869e-5],
f([1.2406358492617064e-12, 6.40000573618869e-5]) = 8.192014367046571e-9
14. x = [2.5600023426720605e-5, 3.84000346795478e-5], f([2.5600023426720605e-5,
3.84000346795478e-5]) = 1.6384029750422933e-9
15. x = [1.0173143286161621e-12, 1.2800014112273815e-5],
f([1.0173143286161621e-12, 1.2800014112273815e-5]) = 3.276806704622707e-10
16. x = [5.120005960429759e-6, 7.680008762232643e-6], f([5.120005960429759e-6,
7.680008762232643e-6]) = 6.553615076014975e-11
17. x = [4.0998192135725505e-13, 2.5600039254326795e-6],
f([4.0998192135725505e-13, 2.5600039254326795e-6]) = 1.3107235998240816e-11
18. x = [1.0240016826511797e-6, 1.5360024887706478e-6],
f([1.0240016826511797e-6, 1.5360024887706478e-6]) = 2.621448543078865e-12
19. x = [6.50489277175567e-14, 5.12001011992864e-7], f([6.50489277175567e-14,
```

```
5.12001011992864e-7]) = 5.242899393429833e-13
20. x = [2.048004222892293e-7, 3.072006287329907e-7], f([2.048004222892293e-7,
3.072006287329907e-7]) = 1.048580304991432e-13
21. x = [6.105355806159245e-15, 1.0240023135252877e-7],
f([6.105355806159245e-15, 1.0240023135252877e-7]) = 2.0971612261343592e-14
22. x = [4.096009413622503e-8, 6.14401408795172e-8], f([4.096009413622503e-8,
6.14401408795172e-8]) = 4.194323252511695e-15
23. x = [5.772500948176087e-16, 2.048004861982361e-8], f([5.772500948176087e-16,
2.048004861982361e-8]) = 8.388647356522391e-16
24. x = [8.192019638207948e-9, 1.2288029327965716e-8], f([8.192019638207948e-9,
1.2288029327965716e-8]) = 1.677729622627484e-16
```

[14]:
```
([8.192019638207948e-9, 1.2288029327965716e-8], [10.0 2.0 3.0; 2.0
3.3384826014781765e-9 1.0000000033384826; … ; 8.388647356522391e-16
5.772500948176087e-16 2.048004861982361e-8; 1.677729622627484e-16
8.192019638207948e-9 1.2288029327965716e-8])
```

[15]:
```
sol, iter = steepestdescent(f1, f1grad, [10.0,10.0], 2.0, true, true)
```

```
0. x = [10.0, 10.0], f([10.0, 10.0]) = 200.0
1. x = [5.000000000465118, 10.0], f([5.000000000465118, 10.0]) = 100.0
2. x = [4.999999999534882, 4.9999999754264985], f([4.999999999534882,
4.9999999754264985]) = 49.999999990697646
3. x = [2.499999987945808, 4.99999998748069], f([2.499999987945808,
4.99999998748069]) = 24.9999998748069
4. x = [2.499999999534882, 2.4999999754264985], f([2.499999999534882,
2.4999999754264985]) = 12.49999999534882
5. x = [1.2499999878295287, 2.49999998748069], f([1.2499999878295287,
2.49999998748069]) = 6.24999993740345
6. x = [1.2499999996511608, 1.2500000252711772], f([1.2499999996511608,
1.2500000252711772]) = 3.1249999982558054
7. x = [0.6249999960013161, 1.2500000124611685], f([0.6249999960013161,
1.2500000124611685]) = 1.562500031152922
8. x = [0.6250000164598511, 0.6250000365727931], f([0.6250000164598511,
0.6250000365727931]) = 0.7812500411496293
9. x = [0.3125000099692602, 0.6250000265163219], f([0.3125000099692602,
0.6250000265163219]) = 0.39062503314540314
10. x = [0.3125000165470615, 0.3125000193296728], f([0.3125000165470615,
0.3125000193296728]) = 0.19531252068382743
11. x = [0.15625000808538766, 0.3125000179383671], f([0.15625000808538766,
0.3125000179383671]) = 0.09765626121147977
12. x = [0.1562500098529794, 0.15625001276559403], f([0.1562500098529794,
0.15625001276559403]) = 0.04882813115811234
13. x = [0.07812500430351281, 0.15625001130928667], f([0.07812500430351281,
0.15625001130928667]) = 0.024414066603415222
14. x = [0.07812500700577368, 0.07812500937494776], f([0.07812500700577368,
0.07812500937494776]) = 0.012207033439304385
15. x = [0.039062503647831766, 0.0781250081903607], f([0.039062503647831766,
```

0.0781250081903607]) = 0.006103516904743926
16. x = [0.03906250454252888, 0.039062505661911356], f([0.03906250454252888, 0.039062505661911356]) = 0.0030517585222701814
17. x = [0.019531252311134607, 0.039062505102220105], f([0.019531252311134607, 0.039062505102220105]) = 0.001525879304860972
18. x = [0.019531252791085474, 0.019531253318174426], f([0.019531252791085474, 0.019531253318174426]) = 0.0007629396711785688
19. x = [0.00976562639917667, 0.01953125305462994], f([0.00976562639917667, 0.01953125305462994]) = 0.00038146984588399154
20. x = [0.009765626655453257, 0.009765626902696594], f([0.009765626655453257, 0.009765626902696594]) = 0.00019073492794739847
21. x = [0.004882813321393021, 0.009765626779074922], f([0.004882813321393021, 0.009765626779074922]) = 9.536746638818527e-5
22. x = [0.004882813457681894, 0.00488281357315299], f([0.004882813457681894, 0.00488281357315299]) = 4.768373452503885e-5
23. x = [0.0024414067215988555, 0.00488281351541744], f([0.0024414067215988555, 0.00488281351541744]) = 2.3841867826343228e-5
24. x = [0.002441406793818581, 0.0024414068474788433], f([0.002441406793818581, 0.0024414068474788433]) = 1.1920934265807051e-5
25. x = [0.001220703391250601, 0.0024414068206487114], f([0.001220703391250601, 0.0024414068206487114]) = 5.96046726391005e-6
26. x = [0.001220703429398108, 0.0012207034704917847], f([0.001220703429398108, 0.0012207034704917847]) = 2.980233725088607e-6
27. x = [0.0006103517190014818, 0.0012207034499449457], f([0.0006103517190014818, 0.0012207034499449457]) = 1.4901169127074928e-6
28. x = [0.0006103517309434633, 0.0006103517504714804], f([0.0006103517309434633, 0.0006103517504714804]) = 7.450584709313643e-7
29. x = [0.00030517586711353465, 0.0006103517407074716], f([0.00030517586711353465, 0.0006103517407074716]) = 3.7252924738464066e-7
30. x = [0.0003051758735939367, 0.0003051758762545648 7], f([0.0003051758735939367, 0.00030517587625456487]) = 1.8626462764764486e-7
31. x = [0.00015258793658485165, 0.00030517587492425076], f([0.00015258793658485165, 0.00030517587492425076]) = 9.313231463578194e-8
32. x = [0.00015258793833939906, 0.0001525879411553668], f([0.00015258793833939906, 0.0001525879411553668]) = 4.656615785333651e-8
33. x = [7.629396854713192e-5, 0.0001525879397473829], f([7.629396854713192e-5, 0.0001525879397473829]) = 2.328307935635096e-8
34. x = [7.629397120025079e-5, 7.629397349970635e-5], f([7.629397120025079e-5, 7.629397349970635e-5]) = 1.1641540083009406e-8
35. x = [3.814698573457742e-5, 7.629397234997854e-5], f([3.814698573457742e-5, 7.629397234997854e-5]) = 5.820770216939291e-9
36. x = [3.814698661540107e-5, 3.814698770145251e-5], f([3.814698661540107e-5, 3.814698770145251e-5]) = 2.910385175671179e-9
37. x = [1.907349334308836e-5, 3.8146987158426776e-5], f([1.907349334308836e-5, 3.8146987158426776e-5]) = 1.4551926292651776e-9
38. x = [1.907349381533839e-5, 1.9073494326525932e-5], f([1.907349381533839e-5, 1.9073494326525932e-5]) = 7.275963326475042e-10
39. x = [9.536746909444012e-6, 1.9073494070932155e-5], f([9.536746909444012e-6,

```
1.9073494070932155e-5]) = 3.637981760738842e-10
40. x = [9.536747161488129e-6, 9.536747401162812e-6], f([9.536747161488129e-6,
9.536747401162812e-6]) = 1.8189909284430388e-10
41. x = [4.7683735736719255e-6, 9.536747281325468e-6], f([4.7683735736719255e-6,
9.536747281325468e-6]) = 9.094954870786873e-11
42. x = [4.7683737076535354e-6, 4.768373819531331e-6], f([4.7683737076535354e-6,
4.768373819531331e-6]) = 4.5474775631683075e-11
43. x = [2.3841868463109237e-6, 4.7683737635924315e-6],
f([2.3841868463109237e-6, 4.7683737635924315e-6]) = 2.2737388349316652e-11
44. x = [2.384186917281503e-6, 2.3841870010788953e-6], f([2.384186917281503e-6,
2.3841870010788953e-6]) = 1.1368694513072568e-11
45. x = [1.1920934688120761e-6, 2.384186959180198e-6], f([1.1920934688120761e-6,
2.384186959180198e-6]) = 5.68434745632492e-12
46. x = [1.1920934903681212e-6, 1.1920935045192326e-6],
f([1.1920934903681212e-6, 1.1920935045192326e-6]) = 2.8421737795561007e-12
47. x = [5.960467363959298e-7, 1.1920934974436767e-6], f([5.960467363959298e-7,
1.1920934974436767e-6]) = 1.4210869066474978e-12
48. x = [5.960467610477448e-7, 5.960467879667743e-7], f([5.960467610477448e-7,
5.960467879667743e-7]) = 7.105434827110162e-13
49. x = [2.9802337809197755e-7, 5.960467745072589e-7], f([2.9802337809197755e-7,
5.960467745072589e-7]) = 3.5527175740050737e-13
50. x = [2.980233964152791e-7, 2.9802341429660983e-7], f([2.980233964152791e-7,
2.9802341429660983e-7]) = 1.776358896217978e-13
51. x = [1.4901169674295845e-7, 2.980234053559438e-7], f([1.4901169674295845e-7,
2.980234053559438e-7]) = 8.881795013995334e-14
52. x = [1.4901170861298351e-7, 1.490117200548806e-7], f([1.4901170861298351e-7,
1.490117200548806e-7]) = 4.440897860752167e-14
53. x = [7.450585482476679e-8, 1.4901171433393165e-7], f([7.450585482476679e-8,
1.4901171433393165e-7]) = 2.2204491008737274e-14
54. x = [7.45058595091643e-8, 7.450586387592613e-8], f([7.45058595091643e-8,
7.450586387592613e-8]) = 1.1102246202398695e-14
55. x = [3.725292933662608e-8, 7.450586169254506e-8], f([3.725292933662608e-8,
7.450586169254506e-8]) = 5.551123426548662e-15
56. x = [3.7252932355918514e-8, 3.7252935164607425e-8],
f([3.7252932355918514e-8, 3.7252935164607425e-8]) = 2.7755619382292965e-15
57. x = [1.862646628163515e-8, 3.725293376026287e-8], f([1.862646628163515e-8,
3.725293376026287e-8]) = 1.3877810737465348e-15
```

[15]:
```
([1.862646628163515e-8, 3.725293376026287e-8], [200.0 10.0 10.0; 100.0
5.000000000465118 10.0; … ; 2.7755619382292965e-15 3.7252932355918514e-8
3.7252935164607425e-8; 1.3877810737465348e-15 1.862646628163515e-8
3.725293376026287e-8])
```

[16]:
```
sol, iter = steepestdescent(f1, f1grad, [100.0,100.0], 2.0, true, true)
```

```
0. x = [100.0, 100.0], f([100.0, 100.0]) = 20000.0
1. x = [50.00000000465118, 100.0], f([50.00000000465118, 100.0]) = 10000.0
2. x = [49.99999999534882, 49.99999975426499], f([49.99999999534882,
```

```
49.99999975426499]) = 4999.999999069764
3. x = [24.999999879458084, 49.9999998748069], f([24.999999879458084,
49.9999998748069]) = 2499.9999874806904
4. x = [24.99999999534882, 24.99999975426499], f([24.99999999534882,
24.99999975426499]) = 1249.9999995348821
5. x = [12.49999987829529, 24.999999874806903], f([12.49999987829529,
24.999999874806903]) = 624.9999937403452
6. x = [12.49999999651161, 12.500000252711775], f([12.49999999651161,
12.500000252711775]) = 312.49999982558063
7. x = [6.249999960013162, 12.50000012461169], f([6.249999960013162,
12.50000012461169]) = 156.2500031152923
8. x = [6.250000164598514, 6.250000365727932], f([6.250000164598514,
6.250000365727932]) = 78.125004114963
9. x = [3.1250000996926026, 6.25000026516322], f([3.1250000996926026,
6.25000026516322]) = 39.06250331454032
10. x = [3.125000165470614, 3.1250002608189593], f([3.125000165470614,
3.1250002608189593]) = 19.531252068382752
11. x = [1.5625000888237965, 3.1250002131447854], f([1.5625000888237965,
3.1250002131447854]) = 9.765626332154953
12. x = [1.5625001243209877, 1.562500135625861], f([1.5625001243209877,
1.562500135625861]) = 4.882813277006204
13. x = [0.7812500599156866, 1.5625001299734242], f([0.7812500599156866,
1.5625001299734242]) = 2.4414066561669676
14. x = [0.7812500700577374, 0.7812500728839603], f([0.7812500700577374,
0.7812500728839603]) = 1.220703343930439
15. x = [0.39062503249335817, 0.7812500714708488], f([0.39062503249335817,
0.7812500714708488]) = 0.6103516741732063
16. x = [0.39062503897749046, 0.3906250441938745], f([0.39062503897749046,
0.3906250441938745]) = 0.30517584215233196
17. x = [0.19531252012262626, 0.39062504158568245], f([0.19531252012262626,
0.39062504158568245]) = 0.15258792311381614
18. x = [0.1953125214630562, 0.1953125191445551], f([0.1953125214630562,
0.1953125191445551]) = 0.07629396208051357
19. x = [0.0976562611933118, 0.19531252030380564], f([0.0976562611933118,
0.19531252030380564]) = 0.03814698058742449
20. x = [0.09765625911049375, 0.09765625729011158], f([0.09765625911049375,
0.09765625729011158]) = 0.019073489886911793
21. x = [0.04882812945557291, 0.09765625820030265], f([0.04882812945557291,
0.09765625820030265]) = 0.009536744765684177
22. x = [0.048828128744729714, 0.048828128156020244], f([0.048828128744729714,
0.048828128156020244]) = 0.0047683723134238016
23. x = [0.02441406448326867, 0.04882812845037497], f([0.02441406448326867,
0.04882812845037497]) = 0.0023841861279663176
24. x = [0.024414063967106287, 0.024414063833492348], f([0.024414063967106287,
0.024414063833492348]) = 0.001192093038779915
25. x = [0.012207031917881715, 0.024414063900299317], f([0.012207031917881715,
0.024414063900299317]) = 0.0005960465161278985
26. x = [0.012207031982417602, 0.01220703201976412], f([0.012207031982417602,
```

0.01220703201976412]) = 0.00029802325963953244
27. x = [0.006103516010449831, 0.012207032001090861], f([0.006103516010449831, 0.012207032001090861]) = 0.0001490116302756564
28. x = [0.0061035159906410305, 0.006103515979885108], f([0.0061035159906410305, 0.006103515979885108]) = 7.450581489602152e-5
29. x = [0.0030517579902264395, 0.006103515985263069], f([0.0030517579902264395, 0.006103515985263069]) = 3.7252907382361815e-5
30. x = [0.00305175799503663, 0.003051757974944078], f([0.00305175799503663, 0.003051757974944078]) = 1.862645372053998e-5
31. x = [0.0015258789876139817, 0.003051757984990354], f([0.0015258789876139817, 0.003051757984990354]) = 9.313226798952386e-6
32. x = [0.0015258789973763723, 0.001525878979972801], f([0.0015258789973763723, 0.001525878979972801]) = 4.656613429268647e-6
33. x = [0.00076293950265073, 0.0015258789886745865], f([0.00076293950265073, 0.0015258789886745865]) = 2.328306688078579e-6
34. x = [0.0007629394860238558, 0.0007629394721569798], f([0.0007629394860238558, 0.0007629394721569798]) = 1.164153318668691e-6
35. x = [0.00038146974241065455, 0.0007629394790904176], f([0.00038146974241065455, 0.0007629394790904176]) = 5.820766487547578e-7
36. x = [0.00038146973667976293, 0.0003814697322578999], f([0.00038146973667976293, 0.0003814697322578999]) = 2.9103832000505534e-7
37. x = [0.00019073486929503222, 0.00038146973446883135], f([0.00019073486929503222, 0.00038146973446883135]) = 1.455191583157207e-7
38. x = [0.00019073486517379897, 0.00019073486167159494], f([0.00019073486517379897, 0.00019073486167159494]) = 7.275957758573457e-8
39. x = [9.536743241883858e-5, 0.0001907348634226969], f([9.536743241883858e-5, 0.0001907348634226969]) = 3.637978812487484e-8
40. x = [9.536743100385829e-5, 9.536742988065e-5], f([9.536743100385829e-5, 9.536742988065e-5]) = 1.8189893792551346e-8
41. x = [4.768371573184554e-5, 9.536743044225413e-5], f([4.768371573184554e-5, 9.536743044225413e-5]) = 9.09494678915818e-9
42. x = [4.768371471040854e-5, 4.768371343244386e-5], f([4.768371471040854e-5, 4.768371343244386e-5]) = 4.547473297167267e-9
43. x = [2.38418571119822e-5, 4.768371407142619e-5], f([2.38418571119822e-5, 4.768371407142619e-5]) = 2.2737365876455284e-9
44. x = [2.3841856959443993e-5, 2.3841856992588765e-5], f([2.3841856959443993e-5, 2.3841856992588765e-5]) = 1.1368682865491758e-9
45. x = [1.192092849740331e-5, 2.384185697601638e-5], f([1.192092849740331e-5, 2.384185697601638e-5]) = 5.684341440648209e-10
46. x = [1.192092847861307e-5, 1.1920928437706593e-5], f([1.192092847861307e-5, 1.1920928437706593e-5]) = 2.8421707158441623e-10
47. x = [5.96046421940776e-6, 1.1920928458159831e-5], f([5.96046421940776e-6, 1.1920928458159831e-5]) = 1.4210853530456493e-10
48. x = [5.960464238752071e-6, 5.960464269155023e-6], f([5.960464238752071e-6, 5.960464269155023e-6]) = 7.105426788288461e-11
49. x = [2.9802321348547432e-6, 5.960464253953547e-6], f([2.9802321348547432e-6, 5.960464253953547e-6]) = 3.552713412265802e-11
50. x = [2.980232119098804e-6, 2.980232119930564e-6], f([2.980232119098804e-6,

```
2.980232119930564e-6]) = 1.7763566967416295e-11
51. x = [1.4901160601038979e-6, 2.980232119514684e-6], f([1.4901160601038979e-6,
2.980232119514684e-6]) = 8.881783486186985e-12
52. x = [1.490116059410786e-6, 1.4901160526418082e-6], f([1.490116059410786e-6,
1.4901160526418082e-6]) = 4.4408917410278585e-12
53. x = [7.450580263902121e-7, 1.490116056262971e-6], f([7.450580263902121e-7,
1.490116056262971e-6]) = 2.220445860427367e-12
54. x = [7.450580296360851e-7, 7.450580226591672e-7], f([7.450580296360851e-7,
7.450580226591672e-7]) = 1.110222935050411e-12
55. x = [3.725290175133376e-7, 7.45058026147626e-7], f([3.725290175133376e-7,
7.45058026147626e-7]) = 5.551114623269968e-13
56. x = [3.7252900863428804e-7, 3.7252899954926263e-7],
f([3.7252900863428804e-7, 3.7252899954926263e-7]) = 2.775557245480911e-13
57. x = [1.862645028665083e-7, 3.725290040917753e-7], f([1.862645028665083e-7,
3.725290040917753e-7]) = 1.3877785888960991e-13
58. x = [1.8626450122526693e-7, 1.8626449790910904e-7],
f([1.8626450122526693e-7, 1.8626449790910904e-7]) = 6.938892883339495e-14
59. x = [9.313225050049298e-8, 1.8626449956718794e-7], f([9.313225050049298e-8,
1.8626449956718794e-7]) = 3.4694463799014965e-14
60. x = [9.313224906669492e-8, 9.313224802179339e-8], f([9.313224906669492e-8,
9.313224802179339e-8]) = 1.7347231632441794e-14
61. x = [4.656612478386591e-8, 9.313224854424415e-8], f([4.656612478386591e-8,
9.313224854424415e-8]) = 8.673615718906866e-15
62. x = [4.656612376037819e-8, 4.656612253835665e-8], f([4.656612376037819e-8,
4.656612253835665e-8]) = 4.3368077641337195e-15
63. x = [2.3283061655662933e-8, 4.656612314936741e-8], f([2.3283061655662933e-8,
4.656612314936741e-8]) = 2.1684038251620514e-15
```

[16]:
```
([2.3283061655662933e-8, 4.656612314936741e-8], [20000.0 100.0 100.0; 10000.0
50.00000000465118 100.0; … ; 4.3368077641337195e-15 4.656612376037819e-8
4.656612253835665e-8; 2.1684038251620514e-15 2.3283061655662933e-8
4.656612314936741e-8])
```

We converge to the solution $(0, 0)$, but the method was quite slow close to the solution.

[17]:
```
sol, iter = steepestdescent(f1, f1grad, [2.0,3.0], 0.1, true, true)
```

```
0. x = [2.0, 3.0], f([2.0, 3.0]) = 10.0
1. x = [1.6000000074048677, 2.6000000074048675], f([1.6000000074048677,
2.6000000074048675]) = 7.120000047391152
2. x = [1.3600000088858413, 2.2000000148097354], f([1.3600000088858413,
2.2000000148097354]) = 5.110400068243261
3. x = [1.1520000115515936, 1.8640000186602665], f([1.1520000115515936,
1.8640000186602665]) = 3.6680960734752444
4. x = [0.9760000130325671, 1.579200021089063], f([0.9760000130325671,
1.579200021089063]) = 2.632852550317856
5. x = [0.8268800138026734, 1.337920022333081], f([0.8268800138026734,
1.337920022333081]) = 1.8897848950901366
```

```
6. x = [0.7005440140325205, 1.1335040227051016], f([0.7005440140325205,
1.1335040227051016]) = 1.3564325694131156
7. x = [0.593510413869969, 0.9603200224420807], f([0.593510413869969,
0.9603200224420807]) = 0.9736078006258554
8. x = [0.5028300934295156, 0.8135961817294126], f([0.5028300934295156,
0.8135961817294126]) = 0.6988273289911094
9. x = [0.4260044927998767, 0.6892897487106355], f([0.4260044927998767,
0.6892897487106355]) = 0.5015979077313479
10. x = [0.360916799249144, 0.5839756482959244], f([0.360916799249144,
0.5839756482959244]) = 0.3600323722366426
11. x = [0.30577362018901844, 0.4947521103289335], f([0.30577362018901844,
0.4947521103289335]) = 0.2584207530781202
12. x = [0.25905556903422955, 0.4191607156723282], f([0.25905556903422955,
0.4191607156723282]) = 0.1854868916553111
13. x = [0.21947540080849198, 0.35511865820264615], f([0.21947540080849198,
0.35511865820264615]) = 0.13313708967308996
14. x = [0.18594254406352279, 0.30086135624940485], f([0.18594254406352279,
0.30086135624940485]) = 0.09556192617405829
15. x = [0.15753305183838784, 0.25489383222601064], f([0.15753305183838784,
0.25489383222601064]) = 0.06859156795840621
16. x = [0.13346414370364953, 0.21594952079190521], f([0.13346414370364953,
0.21594952079190521]) = 0.0492330301758803
17. x = [0.11307263743498301, 0.18295537056739625], f([0.11307263743498301,
0.18295537056739625]) = 0.03533803545311841
18. x = [0.09579667603377062, 0.15500227783190335], f([0.09579667603377062,
0.15500227783190335]) = 0.02536461284679652
19. x = [0.08116024661046753, 0.1313200375510599], f([0.08116024661046753,
0.1313200375510599]) = 0.018205980514151676
20. x = [0.06876006457207173, 0.11125612154624956], f([0.06876006457207173,
0.11125612154624956]) = 0.013067722676616083
21. x = [0.05825446172739567, 0.09425769907125611], f([0.05825446172739567,
0.09425769907125611]) = 0.009379630820773909
22. x = [0.04935397213874895, 0.07985640440031114], f([0.04935397213874895,
0.07985640440031114]) = 0.006732425879487195
23. x = [0.0418133563274674, 0.06765543172155274], f([0.0418133563274674,
0.06765543172155274]) = 0.004832339256082697
24. x = [0.035424844072379794, 0.05731860175527575], f([0.035424844072379794,
0.05731860175527575]) = 0.003468512406059575
25. x = [0.030012409616782164, 0.048561098844237746], f([0.030012409616782164,
0.048561098844237746]) = 0.002489597206124491
26. x = [0.02542692154593886, 0.041141623290606104], f([0.02542692154593886,
0.041141623290606104]) = 0.001786960380454355
27. x = [0.021542033697347918, 0.03485574270910449], f([0.021542033697347918,
0.03485574270910449]) = 0.0012826281269348026
28. x = [0.018250703884041035, 0.029530259202988116], f([0.018250703884041035,
0.029530259202988116]) = 0.0009206331209122758
29. x = [0.015462244509623885, 0.0250184371589329], f([0.015462244509623885,
0.0250184371589329]) = 0.0006608036464522032
```

30. x = [0.013099823809231468, 0.021195960169971634], f([0.013099823809231468, 0.021195960169971634]) = 0.00047430561560921335
31. x = [0.011098348866886591, 0.017957505685626388], f([0.011098348866886591, 0.017957505685626388]) = 0.00034044275967037504
32. x = [0.009402672079018529, 0.015213843008921618], f([0.009402672079018529, 0.015213843008921618]) = 0.0002443598996042107
33. x = [0.007966071645966943, 0.012889374679991334], f([0.007966071645966943, 0.012889374679991334]) = 0.00017539442046702453
34. x = [0.006748964223721221, 0.010920053502837986], f([0.006748964223721221, 0.010920053502837986]) = 0.00012589300773486358
35. x = [0.005717814264968261, 0.009251617822077644], f([0.005717814264968261, 0.009251617822077644]) = 9.036233509782691e-5
36. x = [0.00484420999799702, 0.007838096425401238], f([0.00484420999799702, 0.007838096425401238]) = 6.485945288977895e-5
37. x = [0.0041040805834613005, 0.006640541876608883], f([0.0041040805834613005, 0.006640541876608883]) = 4.655422665435992e-5
38. x = [0.0034770328789438267, 0.00562595737813201], f([0.0034770328789438267, 0.00562595737813201]) = 3.341526828893516e-5
39. x = [0.002945789536876036, 0.004766387594369238], f([0.002945789536876036, 0.004766387594369238]) = 2.3984506565032704e-5
40. x = [0.0024957129534547966, 0.004038148384853245], f([0.0024957129534547966, 0.004038148384853245]) = 1.7215380412150762e-5
41. x = [0.0021144019516911514, 0.003421174223715396], f([0.0021144019516911514, 0.003421174223715396]) = 1.235669876849351e-5
42. x = [0.0017913500858047797, 0.002898465324582174], f([0.0017913500858047797, 0.002898465324582174]) = 8.869278563691836e-6
43. x = [0.001517656152060494, 0.002455619237269258], f([0.001517656152060494, 0.002455619237269258]) = 6.366109890202818e-6
44. x = [0.0012857789296123524, 0.0020804340101312447], f([0.0012857789296123524, 0.0020804340101312447]) = 4.569408305659147e-6
45. x = [0.0010893293936116754, 0.0017625719838080035], f([0.0010893293936116754, 0.0017625719838080035]) = 3.279788226081283e-6
46. x = [0.0009228946753266042, 0.0014932749527147445], f([0.0009228946753266042, 0.0014932749527147445]) = 2.3541364851591403e-6
47. x = [0.0007818889187615411, 0.0012651228459830788], f([0.0007818889187615411, 0.0012651228459830788]) = 1.6897306194000845e-6
48. x = [0.0006624269243570404, 0.001071829278672747], f([0.0006624269243570404, 0.001071829278672747]) = 1.2128394356647428e-6
49. x = [0.0005612170982141204, 0.0009080683399780346], f([0.0005612170982141204, 0.0009080683399780346]) = 8.705408304820937e-7
50. x = [0.0004754707572213887, 0.0007693278458408564], f([0.0004754707572213887, 0.0007693278458408564]) = 6.24848859009181e-7
51. x = [0.00040282529112544547, 0.0006517850125690422], f([0.00040282529112544547, 0.0006517850125690422]) = 4.484983161431463e-7
52. x = [0.00034127906439206015, 0.0005522011258351173], f([0.00034127906439206015, 0.0005522011258351173]) = 3.219190315914174e-7
53. x = [0.0002891362641777353, 0.0004678323028197444], f([0.0002891362641777353, 0.0004678323028197444]) = 2.3106410697800706e-7

54. x = [0.00024496017478124006, 0.00039635388868616123],
f([0.00024496017478124006, 0.00039635388868616123]) = 1.6585108767756158e-7
55. x = [0.00020753359112355975, 0.00033579640424524315],
f([0.00020753359112355975, 0.00033579640424524315]) = 1.190430813490575e-7
56. x = [0.00017582528050979882, 0.00028449127994633895],
f([0.00017582528050979882, 0.00028449127994633895]) = 8.544565739978315e-8
57. x = [0.00014896156857780094, 0.00024102488097638023],
f([0.00014896156857780094, 0.00024102488097638023]) = 6.13304048059146e-8
58. x = [0.00012620226652743633, 0.00020419955669866515],
f([0.00012620226652743633, 0.00020419955669866515]) = 4.40211787014339e-8
59. x = [0.0001069202763419048, 0.00017300064120773324],
f([0.0001069202763419048, 0.00017300064120773324]) = 3.1597120227660606e-8
60. x = [9.05843120538884e-5, 0.00014656849575071822], f([9.05843120538884e-5, 0.00014656849575071822]) = 2.2679492828953233e-8
61. x = [7.674426096727434e-5, 0.00012417482268654175], f([7.674426096727434e-5, 0.00012417482268654175]) = 1.6278679552836683e-8
62. x = [6.501878148513563e-5, 0.00010520259835005182], f([6.501878148513563e-5, 0.00010520259835005182]) = 1.1684362167289879e-8
63. x = [5.508479582094748e-5, 8.91290719016412e-5], f([5.508479582094748e-5, 8.91290719016412e-5]) = 8.386694928987934e-9
64. x = [4.666858808064824e-5, 7.551136172145707e-5], f([4.666858808064824e-5, 7.551136172145707e-5]) = 6.0197254094723194e-9
65. x = [3.9538262436710275e-5, 6.397425247871046e-5], f([3.9538262436710275e-5, 6.397425247871046e-5]) = 4.320783611693814e-9
66. x = [3.3497353590656566e-5, 5.419985664285566e-5], f([3.3497353590656566e-5, 5.419985664285566e-5]) = 3.10133266040757e-9
67. x = [2.8379413470017753e-5, 4.5918855575275317e-5],
f([2.8379413470017753e-5, 4.5918855575275317e-5]) = 2.2260462765318142e-9
68. x = [2.404342500438223e-5, 3.890307886304954e-5], f([2.404342500438223e-5, 3.890307886304954e-5]) = 1.5977911974815178e-9
69. x = [2.0369916614100872e-5, 3.2959217429616386e-5],
f([2.0369916614100872e-5, 3.2959217429616386e-5]) = 1.1468479957779246e-9
70. x = [1.725767035228116e-5, 2.7923497196632285e-5], f([1.725767035228116e-5, 2.7923497196632285e-5]) = 8.231740965233705e-10
71. x = [1.4620932997920874e-5, 2.365716653787087e-5], f([1.4620932997920874e-5, 2.365716653787087e-5]) = 5.908503966364175e-10
72. x = [1.2387053256086483e-5, 2.0042673188802983e-5],
f([1.2387053256086483e-5, 2.0042673188802983e-5]) = 4.240952098466578e-10
73. x = [1.0494479961774128e-5, 1.6980425272405236e-5],
f([1.0494479961774128e-5, 1.6980425272405236e-5]) = 3.0440319248115275e-10
74. x = [8.891066130999589e-6, 1.4386047196180358e-5], f([8.891066130999589e-6, 1.4386047196180358e-5]) = 2.184917477050064e-10
75. x = [7.532632129819622e-6, 1.2188054810797657e-5], f([7.532632129819622e-6, 1.2188054810797657e-5]) = 1.568270142835112e-10
76. x = [6.381748371588343e-6, 1.0325885772879232e-5], f([6.381748371588343e-6, 1.0325885772879232e-5]) = 1.1256586423706423e-10
77. x = [5.406704001519548e-6, 8.748230841568691e-6], f([5.406704001519548e-6, 8.748230841568691e-6]) = 8.079649956563261e-11

78. x = [4.5806331522237495e-6, 7.411620130292598e-6], f([4.5806331522237495e-6, 7.411620130292598e-6]) = 5.799337469049334e-11
79. x = [3.8807746955176875e-6, 6.279225360028142e-6], f([3.8807746955176875e-6, 6.279225360028142e-6]) = 4.1625955654922086e-11
80. x = [3.2878450940912076e-6, 5.31984511198417e-6], f([3.2878450940912076e-6, 5.31984511198417e-6]) = 2.9877898870913244e-11
81. x = [2.7855070729112762e-6, 4.5070451198736765e-6], f([2.7855070729112762e-6, 4.5070451198736765e-6]) = 2.1445485800755723e-11
82. x = [2.359919470410276e-6, 3.818429913836477e-6], f([2.359919470410276e-6, 3.818429913836477e-6]) = 1.539294524080962e-11
83. x = [1.9993558662914607e-6, 3.2350257472660737e-6], f([1.9993558662914607e-6, 3.2350257472660737e-6]) = 1.1048607869643784e-11
84. x = [1.6938814778197153e-6, 2.7407578040262003e-6], f([1.6938814778197153e-6, 2.7407578040262003e-6]) = 7.930369006544579e-12
85. x = [1.4350794219654107e-6, 2.322007281295587e-6], f([1.4350794219654107e-6, 2.322007281295587e-6]) = 5.692187950009173e-12
86. x = [1.2158188009703066e-6, 1.9672361441310998e-6], f([1.2158188009703066e-6, 1.9672361441310998e-6]) = 4.0856867607913985e-12
87. x = [1.0300582212853326e-6, 1.6666692124309284e-6], f([1.0300582212853326e-6, 1.6666692124309284e-6]) = 2.932586986569762e-12
88. x = [8.726793321428626e-7, 1.4120248206867102e-6], f([8.726793321428626e-7, 1.4120248206867102e-6]) = 2.104925545523829e-12
89. x = [7.393457971715496e-7, 1.196286629262953e-6], f([7.393457971715496e-7, 1.196286629262953e-6]) = 1.5108542636552369e-12
90. x = [6.263838132306625e-7, 1.0135102998099781e-6], f([6.263838132306625e-7, 1.0135102998099781e-6]) = 1.084447196177261e-12
91. x = [5.306808843417926e-7, 8.586597080448723e-7], f([5.306808843417926e-7, 8.586597080448723e-7]) = 7.783846196068857e-13
92. x = [4.4960006158728935e-7, 7.274681809922803e-7], f([4.4960006158728935e-7, 7.274681809922803e-7]) = 5.587018143219213e-13
93. x = [3.8090728598602225e-7, 6.163209352878605e-7], f([3.8090728598602225e-7, 6.163209352878605e-7]) = 4.010198935896925e-13
94. x = [3.227098323897098e-7, 5.221554773103321e-7], f([3.227098323897098e-7, 5.221554773103321e-7]) = 2.878404023975951e-13
95. x = [2.734041583148298e-7, 4.423772208189518e-7], f([2.734041583148298e-7, 4.423772208189518e-7]) = 2.0660345927172488e-13
96. x = [2.316317207638451e-7, 3.7478799706852615e-7], f([2.316317207638451e-7, 3.7478799706852615e-7]) = 1.482939470188842e-13
97. x = [1.9624154363532845e-7, 3.17525487606707e-7], f([1.9624154363532845e-7, 3.17525487606707e-7]) = 1.0644107702725802e-13
98. x = [1.6625850432479958e-7, 2.690119109162471e-7], f([1.6625850432479958e-7, 2.690119109162471e-7]) = 7.640030565293347e-14
99. x = [1.4085646570170562e-7, 2.2791054904054348e-7], f([1.4085646570170562e-7, 2.2791054904054348e-7]) = 5.483791471188219e-14
100. x = [1.1933551315495804e-7, 1.930889163496323e-7], f([1.1933551315495804e-7, 1.930889163496323e-7]) = 3.936105836550671e-14
101. x = [1.0110266950837403e-7, 1.635875556178968e-7], f([1.0110266950837403e-7, 1.635875556178968e-7]) = 2.8252221547679072e-14

```
102. x = [8.565555643479309e-8, 1.3859360163678e-7], f([8.565555643479309e-8,
1.3859360163678e-7]) = 2.0278621955923266e-14
103. x = [7.256855218393946e-8, 1.1741838394798446e-7], f([7.256855218393946e-8,
1.1741838394798446e-7]) = 1.455540435067257e-14
104. x = [6.148106422123513e-8, 9.947845157447353e-8], f([6.148106422123513e-8,
9.947845157447353e-8]) = 1.0447445406895359e-14
105. x = [5.2087593647929954e-8, 8.42794969145438e-8], f([5.2087593647929954e-8,
8.42794969145438e-8]) = 7.498872095917792e-15
106. x = [4.4129317642728464e-8, 7.140273584627505e-8],
f([4.4129317642728464e-8, 7.140273584627505e-8]) = 5.3824720322367446e-15
107. x = [3.738695799187142e-8, 6.049336876681246e-8], f([3.738695799187142e-8,
6.049336876681246e-8]) = 3.86338169357254e-15
108. x = [3.1674739210844834e-8, 5.125080462793596e-8],
f([3.1674739210844834e-8, 5.125080462793596e-8]) = 2.773022882578525e-15
```

[17]:
```
([3.1674739210844834e-8, 5.125080462793596e-8], [10.0 2.0 3.0; 7.120000047391152
1.6000000074048677 2.6000000074048675; … ; 3.86338169357254e-15
3.738695799187142e-8 6.049336876681246e-8; 2.773022882578525e-15
3.1674739210844834e-8 5.125080462793596e-8])
```

[18]:
```julia
sol, iter = steepestdescent_convex(f1, f1grad, [2.0,3.0], 0.1, true, true)
```

```
0. x = [2.0, 3.0], f([2.0, 3.0]) = 10.0
1. x = [-6.864689705565752e-9, 0.9999999931353103], f([-6.864689705565752e-9,
0.9999999931353103]) = 2.0
2. x = [0.39999999550607185, 0.5999999935104245], f([0.39999999550607185,
0.5999999935104245]) = 0.3999999912131972
3. x = [-4.951900556271482e-10, 0.19999999650389555], f([-4.951900556271482e-10,
0.19999999650389555]) = 0.0799999975992685
4. x = [0.07999999835247451, 0.11999999785430701], f([0.07999999835247451,
0.11999999785430701]) = 0.015999999393085048
5. x = [-6.581302169905712e-10, 0.03999999754132129], f([-6.581302169905712e-10,
0.03999999754132129]) = 0.0031999997119122486
6. x = [0.015999998860773402, 0.023999998285669757], f([0.015999998860773402,
0.023999998285669757]) = 0.0006399999086861844
7. x = [-4.1769140229908075e-11, 0.007999999405088599],
f([-4.1769140229908075e-11, 0.007999999405088599]) = 0.00012799998229944828
8. x = [0.0031999997458548846, 0.00479999963417223], f([0.0031999997458548846,
0.00479999963417223]) = 2.5599996032173687e-5
9. x = [-1.9768119866730993e-11, 0.001599999806989613],
f([-1.9768119866730993e-11, 0.001599999806989613]) = 5.1199988912495524e-6
10. x = [0.0006399999175120787, 0.0009599998776166624],
f([0.0006399999175120787, 0.0009599998776166624]) = 1.0239997377648056e-6
11. x = [-7.647698988845486e-12, 0.0003199999470627073],
f([-7.647698988845486e-12, 0.0003199999470627073]) = 2.0479994202932426e-7
12. x = [0.00012799997707655522, 0.00019199996539753266],
f([0.00012799997707655522, 0.00019199996539753266]) = 4.095998527336782e-8
13. x = [-6.549525461904682e-13, 6.39999885352256e-5],
```

```
f([-6.549525461904682e-13, 6.39999885352256e-5]) = 8.191997232685837e-9
14. x = [2.5599995215294852e-5, 3.8399992926959216e-5],
f([2.5599995215294852e-5, 3.8399992926959216e-5]) = 1.6383993928834648e-9
15. x = [-1.9115841925327302e-13, 1.2799997104438184e-5],
f([-1.9115841925327302e-13, 1.2799997104438184e-5]) = 3.276798615345608e-10
16. x = [5.119998791864238e-6, 7.679998197878894e-6], f([5.119998791864238e-6,
7.679998197878894e-6]) = 6.553596917497331e-11
17. x = [1.4270822953801714e-14, 2.5599993799553316e-6],
f([1.4270822953801714e-14, 2.5599993799553316e-6]) = 1.3107193504610174e-11
18. x = [1.0239997515401352e-6, 1.5359996369776903e-6],
f([1.0239997515401352e-6, 1.5359996369776903e-6]) = 2.621438747684656e-12
19. x = [-2.0586501567096513e-14, 5.119998261811024e-7],
f([-2.0586501567096513e-14, 5.119998261811024e-7]) = 5.242876861801006e-13
20. x = [2.0479992620371114e-7, 3.071998876254902e-7], f([2.0479992620371114e-7,
3.071998876254902e-7]) = 1.0485752374445474e-13
21. x = [4.028915548807309e-15, 1.0239997217100033e-7],
f([4.028915548807309e-15, 1.0239997217100033e-7]) = 2.0971506950999988e-14
22. x = [4.0959990360515646e-8, 6.143998422783399e-8], f([4.0959990360515646e-8,
6.143998422783399e-8]) = 4.194301918277864e-15
23. x = [2.667964187437204e-15, 2.0480001787039767e-8],
f([2.667964187437204e-15, 2.0480001787039767e-8]) = 8.388607278346872e-16
24. x = [8.192001380727594e-9, 1.228800200709067e-8], f([8.192001380727594e-9,
1.228800200709067e-8]) = 1.6777221550601903e-16
```

[18]:
```
([8.192001380727594e-9, 1.228800200709067e-8], [10.0 2.0 3.0; 2.0
-6.864689705565752e-9 0.9999999931353103; … ; 8.388607278346872e-16
2.667964187437204e-15 2.0480001787039767e-8; 1.6777221550601903e-16
8.192001380727594e-9 1.228800200709067e-8])
```

[19]:
```
k = [x = i for i=1:length(iter[:,1])]
Plots.plot(k,iter[:,1])
```

[19]:

[ ]:
```
k
```

[ ]:
```
k = [x = i for i=10:length(iter[:,1])]
Plots.plot(k,iter[10:length(iter[:,1]),1])
```

1.2 Coordinate descent

[ ]:
```
function Jacobi(f::Function, x0, h::Float64, verbose::Bool = true,  ::Float64 =␣
 ↪1e-6, maxiter::Int64 = 1000)

    function fsearch( ::Float64)
        return(f(x0- *d))
    end
```

```
    x = copy(x0)
    n = length(x)
    k = 0
    d = zeros(n)

    while true
        x0[:] = x[:]
        k += 1

        for i = 1:n
            d[i] = 1.0  # d is now the i^th vector of the canonical basis
              = Optim.minimizer(optimize(fsearch, 0, h, GoldenSection()))
            x[i] -=
            d[i] = 0.0
        end

        if verbose
            println(k, ". ", f(x), " ", x, " ", x0)
        end

        if norm(x-x0) <
            break
        end
    end

    return x
end
```

[ ]:

```
sol = Jacobi(f1, [2.0,3.0], 1.0)
```

1.3 Exemple 2

Consider the bivariate function

$$f(x,y) = \frac{(2-x)^2}{2y^2} + \frac{(3-x)^2}{2y^2} + \ln y$$

that is computed in Julia as

[ ]:

```
f(x) = (2-x[1])*(2-x[1])/(2*x[2]*x[2])+(3-x[1])*(3-x[1])/(2*x[2]*x[2])+log(x[2])
```

Its derivative is

$$\nabla f(x) = \begin{pmatrix} \frac{-2(2-x)}{2y^2} + \frac{-2(3-x)}{2y^2} \\ -\frac{(2-x)^2}{y^3} - \frac{(3-x)^2}{y^3} + \frac{1}{y} \end{pmatrix} = \begin{pmatrix} \frac{x-2}{y^2} + \frac{x-3}{y^2} \\ -\frac{(2-x)^2}{y^3} - \frac{(3-x)^2}{y^3} + \frac{1}{y} \end{pmatrix}$$

```
[ ]: function fprime(x)
         return [(x[1]-2)/(x[2]*x[2])+(x[1]-3)/(x[2]*x[2]),
                 -(2-x[1])*(2-x[1])/(x[2]*x[2]*x[2])-(3-x[1])*(3-x[1])/
      ↪(x[2]*x[2]*x[2])+1/x[2]]
     end
```

```
[ ]: default(size=(600,600), fc=:heat)
     x, y = -2.5:0.1:2.5, 0.5:0.1:2.5
     z = Surface((x,y)->f([x,y]), x, y)
     surface(x,y,z, linealpha = 0.3)
```

```
[ ]: sol = steepestdescent(f, fprime, [1.0,1.0], 2.0)
```

The choice of h is important. Consider for instance a too small value: $h = 0.1$.

```
[ ]: sol = steepestdescent(f, fprime, [1.0,1.0], 0.1)
```

But a too big h can lead to some issues too. Consider for instance $h = 10$.

```
[ ]: sol = steepestdescent(f, fprime, [1.0,1.0], 10.0)
```

We will have to ensure that the iterates are such that $y > 0$ due to the logarithmic operator.

The choice of the starting point is also important to ensure that the algorithm converges fast enough. Consider for instance $x_0 = (0.1, 0.1)$.

```
[ ]: sol = steepestdescent(f, fprime, [0.1,0.1], 2.0)
```

Now, take $x_0 = (100, 100)$.

```
[ ]: sol = steepestdescent(f, fprime, [100.0,100.0], 5.0)
```

In practice, we often need some insight on the function to optimize in order to be efficient.

1.4 Rosenbrock function

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2$$

$$\nabla f(x, y) = \begin{pmatrix} -2(1-x) - 400x(y - x^2) \\ 200(y - x^2) \end{pmatrix}$$

$$\nabla^2 f(x, y) = \begin{pmatrix} 2 - 400(y - x^2) + 800x^2 & -400x \\ -400x & 200 \end{pmatrix} = \begin{pmatrix} 2 - 400y + 1200x^2 & -400x \\ -400x & 200 \end{pmatrix}$$

```
[23]: function rosenbrock(x::Vector)
        return (1.0 - x[1])^2 + 100.0 * (x[2] - x[1]^2)^2
```

```julia
end

function rosenbrock_gradient(x::Vector)
  return [-2.0 * (1.0 - x[1]) - 400.0 * (x[2] - x[1]^2) * x[1],
          200.0 * (x[2] - x[1]^2)]
end

function rosenbrock_hessian(x::Vector)
  h = zeros(2, 2)
  h[1, 1] = 2.0 - 400.0 * x[2] + 1200.0 * x[1]^2
  h[1, 2] = -400.0 * x[1]
  h[2, 1] = -400.0 * x[1]
  h[2, 2] = 200.0
  return h
end
```

[23]:
```
rosenbrock_hessian (generic function with 1 method)
```

[24]:
```julia
default(size=(600,600))
x, y = 0:0.01:1.0, 0:0.01:1.0
z = Surface((x,y)->rosenbrock([x,y]), x, y)
surface(x,y,z, linealpha = 0.3)
```

[25]:
```julia
Plots.contour(x,y,z, linealpha = 0.1, levels=2500)
```

[26]:
```julia
sol, iter = steepestdescent(rosenbrock, rosenbrock_gradient, [0.0,0.0], 10.0,␣
↪true, true)
```

```
0. x = [0.0, 0.0], f([0.0, 0.0]) = 1.0
1. x = [0.1612620227407261, 0.0], f([0.1612620227407261, 0.0]) =
0.7711096853441531
2. x = [0.16126202280695034, 0.02600544024872085], f([0.16126202280695034,
0.02600544024872085]) = 0.7034813943858886
3. x = [0.21133888041724883, 0.02600543876247529], f([0.21133888041724883,
0.02600543876247529]) = 0.6568010089605405
4. x = [0.21133888100318554, 0.04466412256994727], f([0.21133888100318554,
0.04466412256994727]) = 0.6219863606173075
5. x = [0.24508247498253186, 0.04466412279984179], f([0.24508247498253186,
0.04466412279984179]) = 0.5936204637173165
6. x = [0.2450824743849232, 0.06006541860916487], f([0.2450824743849232,
0.06006541860916487]) = 0.5699004704807902
7. x = [0.2711222718134955, 0.06006542082182948], f([0.2711222718134955,
0.06006542082182948]) = 0.5493311173278929
8. x = [0.2711222708981266, 0.0735072854577914], f([0.2711222708981266,
0.0735072854577914]) = 0.5312627439807038
9. x = [0.29257001561271645, 0.07350728639695933], f([0.29257001561271645,
0.07350728639695933]) = 0.5150738178410098
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10. x = [0.2925700148205872, 0.08559721330212781], f([0.2925700148205872, 0.08559721330212781]) = 0.5004571839309442
11. x = [0.31093080030671383, 0.08559721400286757], f([0.31093080030671383, 0.08559721400286757]) = 0.48709466086751985
12. x = [0.31093080025786574, 0.09667796239376748], f([0.31093080025786574, 0.09667796239376748]) = 0.4748163620332654
13. x = [0.32705633895121194, 0.0966779627570338], f([0.32705633895121194, 0.0966779627570338]) = 0.4634372309681655
14. x = [0.3270563388406716, 0.10696584867584466], f([0.3270563388406716, 0.10696584867584466]) = 0.45285317109452106
15. x = [0.34147917966047076, 0.10696584889021117], f([0.34147917966047076, 0.10696584889021117]) = 0.44294683674908686
16. x = [0.34147917957615054, 0.11660802998007845], f([0.34147917957615054, 0.11660802998007845]) = 0.43364967093169987
17. x = [0.35455629129726013, 0.11660803018645081], f([0.35455629129726013, 0.11660803018645081]) = 0.4248824645510051
18. x = [0.3545562909824612, 0.12571016314624006], f([0.3545562909824612, 0.12571016314624006]) = 0.4165975815103173
19. x = [0.3665395702808029, 0.1257101637570594], f([0.3665395702808029, 0.1257101637570594]) = 0.4087389645403241
20. x = [0.36653956967664153, 0.13435125605737955], f([0.36653956967664153, 0.13435125605737955]) = 0.40127211678545444
21. x = [0.3776139464389042, 0.13435125619961247], f([0.3776139464389042, 0.13435125619961247]) = 0.3941558676722247
22. x = [0.37761394646013513, 0.14259229299276238], f([0.37761394646013513, 0.14259229299276238]) = 0.38736439964092756
23. x = [0.38791953096168935, 0.14259229227816972], f([0.38791953096168935, 0.14259229227816972]) = 0.3808665590438894
24. x = [0.38791953164271525, 0.15048156418366151], f([0.38791953164271525, 0.15048156418366151]) = 0.3746424997444731
25. x = [0.3975652307132169, 0.15048156236546914], f([0.3975652307132169, 0.15048156236546914]) = 0.3686680627004397
26. x = [0.3975652324647904, 0.1580581152372333], f([0.3975652324647904, 0.1580581152372333]) = 0.36292764913520204
27. x = [0.4066375219733344, 0.1580581134715968], f([0.4066375219733344, 0.1580581134715968]) = 0.3574021347367799
28. x = [0.4066375239092683, 0.16535407718962844], f([0.4066375239092683, 0.16535407718962844]) = 0.3520790280325243
29. x = [0.4152063101623946, 0.16535407525659443], f([0.4152063101623946, 0.16535407525659443]) = 0.34694292443681346
30. x = [0.4152063113680399, 0.17239628238293805], f([0.4152063113680399, 0.17239628238293805]) = 0.34198365826377397
31. x = [0.42332897810407716, 0.1723962804618621], f([0.42332897810407716, 0.1723962804618621]) = 0.33718863471913035
32. x = [0.4233289797073569, 0.17920742533977313], f([0.4233289797073569, 0.17920742533977313]) = 0.3325494656453579
33. x = [0.43105326682885253, 0.17920742496512393], f([0.43105326682885253, 0.17920742496512393]) = 0.32805571713162346

34. x = [0.4310532673689011, 0.18580692043008465], f([0.4310532673689011, 0.18580692043008465]) = 0.3237003845716032
35. x = [0.43841934676855787, 0.18580691897915594], f([0.43841934676855787, 0.18580691897915594]) = 0.3194747261456466
36. x = [0.4384193481341236, 0.19221152544167294], f([0.4384193481341236, 0.19221152544167294]) = 0.3153728285501027
37. x = [0.44546138112247474, 0.192211524660468], f([0.44546138112247474, 0.192211524660468]) = 0.31138729255049824
38. x = [0.44546138124991075, 0.19843584240960657], f([0.44546138124991075, 0.19843584240960657]) = 0.3075130796852568
39. x = [0.4522086741038411, 0.19843584213641413], f([0.4522086741038411, 0.19843584213641413]) = 0.3037438711954518
40. x = [0.4522086736606174, 0.2044926845672846], f([0.4522086736606174, 0.2044926845672846]) = 0.3000753372126599
41. x = [0.45868657490787185, 0.20449268452779948], f([0.45868657490787185, 0.20449268452779948]) = 0.29650203781054824
42. x = [0.4586865750709403, 0.21039337432120128], f([0.4586865750709403, 0.21039337432120128]) = 0.2930202240084287
43. x = [0.46491717315029296, 0.21039337412450212], f([0.46491717315029296, 0.21039337412450212]) = 0.2896251780393302
44. x = [0.4649171731882241, 0.21614797820261164], f([0.4649171731882241, 0.21614797820261164]) = 0.28631363154888095
45. x = [0.4709198394704101, 0.21614797789155066], f([0.4709198394704101, 0.21614797789155066]) = 0.28308146634477055
46. x = [0.47091984037103257, 0.221765496557067], f([0.47091984037103257, 0.221765496557067]) = 0.2799258153130136
47. x = [0.47671167009298804, 0.2217654960075416], f([0.47671167009298804, 0.2217654960075416]) = 0.2768430618298368
48. x = [0.4767116699440883, 0.22725401606985668], f([0.4767116699440883, 0.22725401606985668]) = 0.27383067637270475
49. x = [0.4823078273070866, 0.2272540162741426], f([0.4823078273070866, 0.2272540162741426]) = 0.2708854656602998
50. x = [0.48230782658078375, 0.23262083933898028], f([0.48230782658078375, 0.23262083933898028]) = 0.2680051864195119
51. x = [0.4877218312985245, 0.2326208395921666], f([0.4877218312985245, 0.2326208395921666]) = 0.2651870048223571
52. x = [0.4877218314779602, 0.23787258538363515], f([0.4877218314779602, 0.23787258538363515]) = 0.2624289219442954
53. x = [0.4929657963259798, 0.23787258488878021], f([0.4929657963259798, 0.23787258488878021]) = 0.2597284112391086
54. x = [0.49296579638249666, 0.2430152766266245], f([0.49296579638249666, 0.2430152766266245]) = 0.2570836836380358
55. x = [0.49805061568825115, 0.24301527640239076], f([0.49805061568825115, 0.24301527640239076]) = 0.25449247698514527
56. x = [0.49805061658483746, 0.24805441699865693], f([0.49805061658483746, 0.24805441699865693]) = 0.2519531835108618
57. x = [0.5029861310596924, 0.24805441668585915], f([0.5029861310596924, 0.24805441668585915]) = 0.2494637697351823

58. x = [0.5029861316510841, 0.25299504887483176], f([0.5029861316510841, 0.25299504887483176]) = 0.24702278533115352
59. x = [0.5077812596453132, 0.25299504864182587], f([0.5077812596453132, 0.25299504864182587]) = 0.24462839564175978
60. x = [0.5077812602631905, 0.2578418093692486], f([0.5077812602631905, 0.2578418093692486]) = 0.24227928774809313
61. x = [0.5124441128776053, 0.25784180833215453], f([0.5124441128776053, 0.25784180833215453]) = 0.23997380066119134
62. x = [0.5124441139738207, 0.2625989713984018], f([0.5124441139738207, 0.2625989713984018]) = 0.23771074199877293
63. x = [0.5169820918539044, 0.26259897004694943], f([0.5169820918539044, 0.26259897004694943]) = 0.2354886031949667
64. x = [0.5169820935504372, 0.2672704863639032], f([0.5169820935504372, 0.2672704863639032]) = 0.23330629795091873
65. x = [0.5214019702389341, 0.2672704851632513], f([0.5214019702389341, 0.2672704851632513]) = 0.23116245210783615
66. x = [0.5214019711872473, 0.27186001652605396], f([0.5214019711872473, 0.27186001652605396]) = 0.22905607318345256
67. x = [0.5257099605380149, 0.27186001565463513], f([0.5257099605380149, 0.27186001565463513]) = 0.22698590577525354
68. x = [0.5257099614420236, 0.27637096453238463], f([0.5257099614420236, 0.27637096453238463]) = 0.22495104067532679
69. x = [0.5299117809322826, 0.27637096367037745], f([0.5299117809322826, 0.27637096367037745]) = 0.2229503280302457
70. x = [0.5299117819054657, 0.28080649789777257], f([0.5299117819054657, 0.28080649789777257]) = 0.22098293279129458
71. x = [0.5340127048210788, 0.2808064967675729], f([0.5340127048210788, 0.2808064967675729]) = 0.21904779912045264
72. x = [0.534012705860308, 0.28516957158251066], f([0.534012705860308, 0.28516957158251066]) = 0.21714415829963205
73. x = [0.5380176065649581, 0.28516957023983347], f([0.5380176065649581, 0.28516957023983347]) = 0.21527103850466994
74. x = [0.5380176081023069, 0.28946294851869525], f([0.5380176081023069, 0.28946294851869525]) = 0.21342773042351404
75. x = [0.5419310032811242, 0.28946294691721963], f([0.5419310032811242, 0.28946294691721963]) = 0.2116133376782171
76. x = [0.5419310047515339, 0.29368921587758573], f([0.5419310047515339, 0.29368921587758573]) = 0.20982720440793967
77. x = [0.5457570865898584, 0.2936892142349757], f([0.5457570865898584, 0.2936892142349757]) = 0.20806850196298415
78. x = [0.5457570885385001, 0.29785080184311163], f([0.5457570885385001, 0.29785080184311163]) = 0.20633662261302047
79. x = [0.5494997569473468, 0.29785080006910425], f([0.5494997569473468, 0.29785080006910425]) = 0.2046307989664715
80. x = [0.5494997594556297, 0.30194998862636097], f([0.5494997594556297, 0.30194998862636097]) = 0.2029504667305364
81. x = [0.5531626494771366, 0.3019499861996965], f([0.5531626494771366, 0.3019499861996965]) = 0.201294913842769

82. x = [0.5531626519044585, 0.3059889222559571], f([0.5531626519044585, 0.3059889222559571]) = 0.19966361565305693
83. x = [0.5567491557152695, 0.30598892001339], f([0.5567491557152695, 0.30598892001339]) = 0.19805591009997486
84. x = [0.5567491587838322, 0.30996962962092295], f([0.5567491587838322, 0.30996962962092295]) = 0.1964713082388418
85. x = [0.5602624494427373, 0.30996962659754534], f([0.5602624494427373, 0.30996962659754534]) = 0.19490919364939663
86. x = [0.5602624531056657, 0.3138940206724136], f([0.5602624531056657, 0.3138940206724136]) = 0.19336911014864872
87. x = [0.5637055025720049, 0.31389401729587413], f([0.5637055025720049, 0.31389401729587413]) = 0.191850482770056
88. x = [0.5637055063787026, 0.31776390254482173], f([0.5637055063787026, 0.31776390254482173]) = 0.1903528851642664
89. x = [0.5670811022041661, 0.31776389896790874], f([0.5670811022041661, 0.31776389896790874]) = 0.18887578013987083
90. x = [0.5670811061399202, 0.32158098456016976], f([0.5670811061399202, 0.32158098456016976]) = 0.1874187686610363
91. x = [0.5703918675998588, 0.3215809817923052], f([0.5703918675998588, 0.3215809817923052]) = 0.1859813483317945
92. x = [0.5703918705066301, 0.325346889032092], f([0.5703918705066301, 0.325346889032092]) = 0.18456314492679302
93. x = [0.5736402603241066, 0.3253468866941144], f([0.5736402603241066, 0.3253468866941144]) = 0.18316368762260327
94. x = [0.5736402632112945, 0.32906315527968033], f([0.5736402632112945, 0.32906315527968033]) = 0.1817826251545356
95. x = [0.5768286022047667, 0.32906315251089274], f([0.5768286022047667, 0.32906315251089274]) = 0.18041951579613916
96. x = [0.5768286048682323, 0.33273124188066033], f([0.5768286048682323, 0.33273124188066033]) = 0.17907402965776728
97. x = [0.579959081464713, 0.3327312400412868], f([0.579959081464713, 0.3327312400412868]) = 0.17774575181160876
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105. x = [0.5919420598412224, 0.34694863174374074], f([0.5919420598412224, 0.34694863174374074]) = 0.16769930519069065

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116. x = [0.6058368274487557, 0.36703826545700524], f([0.6058368274487557, 0.36703826545700524]) = 0.1553646065956636
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119. x = [0.6110952522253774, 0.37025537955508764], f([0.6110952522253774, 0.37025537955508764]) = 0.15225943289374383
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144. x = [0.6396514215132041, 0.40915394819552947], f([0.6396514215132041, 0.40915394819552947]) = 0.12985109801745962
145. x = [0.6418287772805903, 0.4091539438742461], f([0.6418287772805903, 0.4091539438742461]) = 0.12906516618282
146. x = [0.641828782666134, 0.41194419288938433], f([0.641828782666134, 0.41194419288938433]) = 0.12828662092642787
147. x = [0.6439785853010245, 0.4119441889095427], f([0.6439785853010245, 0.4119441889095427]) = 0.12751534415111143
148. x = [0.6439785903495242, 0.4147084309118433], f([0.6439785903495242, 0.4147084309118433]) = 0.12675124412951558
149. x = [0.6461014981647627, 0.4147084272844714], f([0.6461014981647627, 0.4147084272844714]) = 0.12599420758357335
150. x = [0.6461015027423092, 0.4174471578906037], f([0.6461015027423092, 0.4174471578906037]) = 0.12524414636125547
151. x = [0.6481981496228112, 0.4174471543094515], f([0.6481981496228112, 0.4174471543094515]) = 0.12450095156893304
152. x = [0.6481981545175277, 0.42016085357537497], f([0.6481981545175277, 0.42016085357537497]) = 0.12376453848487697
153. x = [0.6502691519779552, 0.420160850010645], f([0.6502691519779552, 0.420160850010645]) = 0.12303480269753236

154. x = [0.6502691568897336, 0.4228499825668948], f([0.6502691568897336, 0.4228499825668948]) = 0.12231166262262155
155. x = [0.6523150945892143, 0.4228499789604677], f([0.6523150945892143, 0.4228499789604677]) = 0.12159501790580171
156. x = [0.6523150994031902, 0.4255149950691268], f([0.6523150994031902, 0.4255149950691268]) = 0.1208847901030173
157. x = [0.6543365463409216, 0.42551499148785416], f([0.6543365463409216, 0.42551499148785416]) = 0.12018088264858433
158. x = [0.6543365507199344, 0.42815632741190185], f([0.6543365507199344, 0.42815632741190185]) = 0.11948322016819589
159. x = [0.6563340554804089, 0.4281563240580113], f([0.6563340554804089, 0.4281563240580113]) = 0.11879170959816232
160. x = [0.6563340601226241, 0.43077440460445826], f([0.6563340601226241, 0.43077440460445826]) = 0.11810627823180392
161. x = [0.6583081534861536, 0.43077440108474635], f([0.6583081534861536, 0.43077440108474635]) = 0.11742683666322522
162. x = [0.6583081578948825, 0.4333696368763831], f([0.6583081578948825, 0.4333696368763831]) = 0.11675331496119229
163. x = [0.6602593518389702, 0.43336963337853174], f([0.6602593518389702, 0.43336963337853174]) = 0.11608562683729319
164. x = [0.6602593564388131, 0.4359424241093249], f([0.6602593564388131, 0.4359424241093249]) = 0.1154237048873735
165. x = [0.6621881461636036, 0.4359424205075047], f([0.6621881461636036, 0.4359424205075047]) = 0.11476746605444571
166. x = [0.6621881510598719, 0.43849315406870876], f([0.6621881510598719, 0.43849315406870876]) = 0.11411684528435234
167. x = [0.6640950156783447, 0.4384931503067066], f([0.6640950156783447, 0.4384931503067066]) = 0.11347176259268925
168. x = [0.6640950204128417, 0.44102220241565265], f([0.6640950204128417, 0.44102220241565265]) = 0.11283215531145314
169. x = [0.6659804226845412, 0.44102219889158867], f([0.6659804226845412, 0.44102219889158867]) = 0.11219794625054509
170. x = [0.6659804276027701, 0.4435299363265524], f([0.6659804276027701, 0.4435299363265524]) = 0.11156907474443238
171. x = [0.6678448162731484, 0.4435299327673596], f([0.6678448162731484, 0.4435299327673596]) = 0.11094546651865433
172. x = [0.6678448207386024, 0.44601671051703257], f([0.6678448207386024, 0.44601671051703257]) = 0.11032706311017468
173. x = [0.6696886290525018, 0.44601670722546144], f([0.6696886290525018, 0.44601670722546144]) = 0.10971379266985916
174. x = [0.6696886339785264, 0.44848287309557905], f([0.6696886339785264, 0.44848287309557905]) = 0.10910559852297627
175. x = [0.6715122825619668, 0.44848286944314225], f([0.6715122825619668, 0.44848286944314225]) = 0.10850241154056714
176. x = [0.6715122879320504, 0.45092875953897815], f([0.6715122879320504, 0.45092875953897815]) = 0.10790417697964062
177. x = [0.6733161842647911, 0.4509287558622842], f([0.6733161842647911, 0.4509287558622842]) = 0.10731082819280631

178. x = [0.6733161890777579, 0.4533546970565887], f([0.6733161890777579, 0.4533546970565887]) = 0.10672231231868354
179. x = [0.6751007277612031, 0.4533546934609088], f([0.6751007277612031, 0.4533546934609088]) = 0.10613856466738773
180. x = [0.6751007318515787, 0.4557610036852685], f([0.6751007318515787, 0.4557610036852685]) = 0.10555953444338283
181. x = [0.6768662947996549, 0.455761000675426], f([0.6768662947996549, 0.455761000675426]) = 0.104985158960591
182. x = [0.6768662987470866, 0.45814799190373307], f([0.6768662987470866, 0.45814799190373307]) = 0.10441538888541015
183. x = [0.6786132571381945, 0.45814798891721614], f([0.6786132571381945, 0.45814798891721614]) = 0.10385016376515015
184. x = [0.6786132612495241, 0.46051596425787694], f([0.6786132612495241, 0.46051596425787694]) = 0.10328943584467011
185. x = [0.6803419752337734, 0.4605159610767037], f([0.6803419752337734, 0.4605159610767037]) = 0.10273314668336907
186. x = [0.6803419793400892, 0.46286521423121435], f([0.6803419793400892, 0.46286521423121435]) = 0.10218125017221483
187. x = [0.6820527980011969, 0.46286521135245934], f([0.6820527980011969, 0.46286521135245934]) = 0.10163368980964116
188. x = [0.6820528019095984, 0.4651960294397877], f([0.6820528019095984, 0.4651960294397877]) = 0.10109042077353943
189. x = [0.6837460645831317, 0.4651960268584414], f([0.6837460645831317, 0.4651960268584414]) = 0.10055138850722391
190. x = [0.6837460678297285, 0.4675086898685005], f([0.6837460678297285, 0.4675086898685005]) = 0.10001654961316082
191. x = [0.6854221036923195, 0.4675086874328462], f([0.6854221036923195, 0.4675086874328462]) = 0.09948585106442367
192. x = [0.6854221066591583, 0.4698034684426367], f([0.6854221066591583, 0.4698034684426367]) = 0.09895925097876372
193. x = [0.6870812353406935, 0.46980346625621416], f([0.6870812353406935, 0.46980346625621416]) = 0.09843669799546491
194. x = [0.6870812383346512, 0.4720806322068961], f([0.6870812383346512, 0.4720806322068961]) = 0.09791815140217709
195. x = [0.6887237708302745, 0.47208063003619], f([0.6887237708302745, 0.47208063003619]) = 0.09740356156668332
196. x = [0.6887237737643778, 0.4743404405328111], f([0.6887237737643778, 0.4743404405328111]) = 0.09689288901949186
197. x = [0.6903500120482545, 0.47434043845110985], f([0.6903500120482545, 0.47434043845110985]) = 0.09638608567427953
198. x = [0.6903500150394596, 0.4765831471612992], f([0.6903500150394596, 0.4765831471612992]) = 0.09588311318606442
199. x = [0.6919602527171902, 0.47658314513514516], f([0.6919602527171902, 0.47658314513514516]) = 0.09538392503901885
200. x = [0.691960255984583, 0.4788089997205048], f([0.691960255984583, 0.4788089997205048]) = 0.09488848389308512
201. x = [0.6935547789832591, 0.4788089977233385], f([0.6935547789832591, 0.4788089977233385]) = 0.09439674485012964

202. x = [0.6935547819746861, 0.4810182398650694], f([0.6935547819746861, 0.4810182398650694]) = 0.09390867165058397
203. x = [0.6951338681157613, 0.4810182376672962], f([0.6951338681157613, 0.4810182376672962]) = 0.09342422052348195
204. x = [0.6951338711887082, 0.48321110289423375], f([0.6951338711887082, 0.48321110289423375]) = 0.09294335649638479
205. x = [0.6966977907877012, 0.48321110083181235], f([0.6966977907877012, 0.48321110083181235]) = 0.09246603712840736
206. x = [0.6966977935898097, 0.4853878197288562], f([0.6966977935898097, 0.4853878197288562]) = 0.09199222841329141
207. x = [0.6982468098218472, 0.48538781761656125], f([0.6982468098218472, 0.48538781761656125]) = 0.09152188904291705
208. x = [0.698246812540826, 0.48754861515951464], f([0.698246812540826, 0.48754861515951464]) = 0.09105498614177296
209. x = [0.6997811812606548, 0.48754861315807624], f([0.6997811812606548, 0.48754861315807624]) = 0.09059147958758924
210. x = [0.6997811843353042, 0.4896937099332446], f([0.6997811843353042, 0.4896937099332446]) = 0.09013133727911415
211. x = [0.7013011552387945, 0.48969370791643946], f([0.7013011552387945, 0.48969370791643946]) = 0.08967452050961035
212. x = [0.7013011584077622, 0.49182331924293266], f([0.7013011584077622, 0.49182331924293266]) = 0.08922099796854675
213. x = [0.702806974899733, 0.4918233169951114], f([0.702806974899733, 0.4918233169951114]) = 0.08877073202295538
214. x = [0.7028069776636184, 0.4939376516848105], f([0.7028069776636184, 0.4939376516848105]) = 0.08832369252543446
215. x = [0.7042988760784883, 0.493937649761093], f([0.7042988760784883, 0.493937649761093]) = 0.08787984271905568
216. x = [0.7042988785075263, 0.4960369137263927], f([0.7042988785075263, 0.4960369137263927]) = 0.0874391532519079
217. x = [0.7057770895182833, 0.4960369119970235], f([0.7057770895182833, 0.4960369119970235]) = 0.08700158842408748
218. x = [0.7057770919012868, 0.4981213062936671], f([0.7057770919012868, 0.4981213062936671]) = 0.08656711965006464
219. x = [0.7072418407455178, 0.49812130487930095], f([0.7072418407455178, 0.49812130487930095]) = 0.08613571241674267
220. x = [0.7072418422341056, 0.5001910260128434], f([0.7072418422341056, 0.5001910260128434]) = 0.08570733893848101
221. x = [0.7086933478679972, 0.5001910247207054], f([0.7086933478679972, 0.5001910247207054]) = 0.08528196532111919
222. x = [0.7086933493877339, 0.5022462660469088], f([0.7086933493877339, 0.5022462660469088]) = 0.08485956469093756
223. x = [0.710131825704645, 0.5022462647726539], f([0.710131825704645, 0.5022462647726539]) = 0.08444010416187585
224. x = [0.7101318272990877, 0.5042872143455732], f([0.7101318272990877, 0.5042872143455732]) = 0.08402355754496638
225. x = [0.7115574830476378, 0.5042872132623546], f([0.7115574830476378, 0.5042872132623546]) = 0.08360989298337967

226. x = [0.7115574841296047, 0.5063140546638409], f([0.7115574841296047, 0.5063140546638409]) = 0.08319908496164344
227. x = [0.712970522653726, 0.5063140539569422], f([0.712970522653726, 0.5063140539569422]) = 0.08279110242468189
228. x = [0.7129705232775303, 0.5083269676373974], f([0.7129705232775303, 0.5083269676373974]) = 0.0823859205075748
229. x = [0.7143711428157129, 0.5083269673569305], f([0.7143711428157129, 0.5083269673569305]) = 0.08198350905892968
230. x = [0.7143711436295627, 0.5103261323457144], f([0.7143711436295627, 0.5103261323457144]) = 0.08158384359148414
231. x = [0.7157595385831298, 0.5103261316189689], f([0.7157595385831298, 0.5103261316189689]) = 0.08118689486589596
232. x = [0.7157595397408909, 0.5123117206797062], f([0.7157595397408909, 0.5123117206797062]) = 0.08079263924831055
233. x = [0.7171358991694793, 0.5123117197356579], f([0.7171358991694793, 0.5123117197356579]) = 0.08040104820102131
234. x = [0.7171359006874746, 0.5142839020539198], f([0.7171359006874746, 0.5142839020539198]) = 0.08001209867988665
235. x = [0.7185004098086362, 0.5142839010895818], f([0.7185004098086362, 0.5142839010895818]) = 0.07962576301052522
236. x = [0.7185004104783294, 0.5162428408238496], f([0.7185004104783294, 0.5162428408238496]) = 0.07924201890086917
237. x = [0.7198532490779065, 0.5162428403594522], f([0.7198532490779065, 0.5162428403594522]) = 0.07886083910723171
238. x = [0.7198532495186465, 0.5181887014852321], f([0.7198532495186465, 0.5181887014852321]) = 0.07848220180526179
239. x = [0.7211945940140063, 0.5181887011775179], f([0.7211945940140063, 0.5181887011775179]) = 0.07810608059751298
240. x = [0.7211945945029418, 0.5201216437037616], f([0.7211945945029418, 0.5201216437037616]) = 0.0777324541343791
241. x = [0.7225246175068922, 0.5201216434349482], f([0.7225246175068922, 0.5201216434349482]) = 0.07736129680883393
242. x = [0.722524617459731, 0.5220418227869834], f([0.722524617459731, 0.5220418227869834]) = 0.07699258791586865
243. x = [0.7238434869080516, 0.5220418228099633], f([0.7238434869080516, 0.5220418228099633]) = 0.07662630233181007
244. x = [0.7238434867426127, 0.5239493930685757], f([0.7238434867426127, 0.5239493930685757]) = 0.07626241981447754
245. x = [0.7251513669728034, 0.5239493931780378], f([0.7251513669728034, 0.5239493931780378]) = 0.0759009159672303
246. x = [0.7251513674138828, 0.5258445065872175], f([0.7251513674138828, 0.5258445065872175]) = 0.07554177083445855
247. x = [0.726448420198613, 0.5258445061506991], f([0.726448420198613, 0.5258445061506991]) = 0.0751849607943713
248. x = [0.726448420287837, 0.5277273073113219], f([0.726448420287837, 0.5277273073113219]) = 0.0748304667630199
249. x = [0.7277348019174287, 0.5277273073241936], f([0.7277348019174287, 0.5277273073241936]) = 0.07447826546671767

250. x = [0.7277348021015578, 0.5295979425865183], f([0.7277348021015578, 0.5295979425865183]) = 0.0741283379866779
251. x = [0.7290106670859768, 0.5295979424006086], f([0.7290106670859768, 0.5295979424006086]) = 0.07378066178703306
252. x = [0.7290106669100368, 0.5314565522665895], f([0.7290106669100368, 0.5314565522665895]) = 0.07343521864854302
253. x = [0.7302761649095796, 0.5314565523609346], f([0.7302761649095796, 0.5314565523609346]) = 0.07309198641808055
254. x = [0.7302761652351798, 0.5333032779537366], f([0.7302761652351798, 0.5333032779537366]) = 0.07275094704024004
255. x = [0.7315314441967632, 0.5333032777475033], f([0.7315314441967632, 0.5333032777475033]) = 0.07241207918423602
256. x = [0.731531444320189, 0.5351382542896581], f([0.731531444320189, 0.5351382542896581]) = 0.07207536538880377
257. x = [0.7327766476739511, 0.5351382541688448], f([0.7327766476739511, 0.5351382541688448]) = 0.07174078463764738
258. x = [0.7327766481710549, 0.5369616172515452], f([0.7327766481710549, 0.5369616172515452]) = 0.07140831976269629
259. x = [0.7340119173859374, 0.5369616167214528], f([0.7340119173859374, 0.5369616167214528]) = 0.07107795033325964
260. x = [0.7340119179122655, 0.5387734967280415], f([0.7340119179122655, 0.5387734967280415]) = 0.07074965981271152
261. x = [0.7352373907359088, 0.5387734962254838], f([0.7352373907359088, 0.5387734962254838]) = 0.07042342811567988
262. x = [0.7352373909421971, 0.5405740211689226], f([0.7352373909421971, 0.5405740211689226]) = 0.07009923915509499
263. x = [0.7364532018398187, 0.5405740211094857], f([0.7364532018398187, 0.5405740211094857]) = 0.06977707333569666
264. x = [0.7364532022366673, 0.5423633196745706], f([0.7364532022366673, 0.5423633196745706]) = 0.06945691461130701
265. x = [0.7376594840689393, 0.542363319404554], f([0.7376594840689393, 0.542363319404554]) = 0.06913874405626466
266. x = [0.7376594843680497, 0.5441415157303816], f([0.7376594843680497, 0.5441415157303816]) = 0.06882254614203762
267. x = [0.7388563661217463, 0.5441415153415604], f([0.7388563661217463, 0.5441415153415604]) = 0.06850830219472998
268. x = [0.7388563672158199, 0.5459087330168845], f([0.7388563672158199, 0.5459087330168845]) = 0.06819599694371895
269. x = [0.740043976322651, 0.5459087322703645], f([0.740043976322651, 0.5459087322703645]) = 0.06788561240163589
270. x = [0.7400439772441988, 0.5476650898029054], f([0.7400439772441988, 0.5476650898029054]) = 0.06757713376701487
271. x = [0.7412224371432343, 0.5476650891013802], f([0.7412224371432343, 0.5476650891013802]) = 0.06727054324145736
272. x = [0.7412224388790682, 0.549410706196096], f([0.7412224388790682, 0.549410706196096]) = 0.06696582613969809
273. x = [0.7423918721338455, 0.5494107051574973], f([0.7423918721338455, 0.5494107051574973]) = 0.06666296541127864

274. x = [0.7423918734337646, 0.5511456954844194], f([0.7423918734337646, 0.5511456954844194]) = 0.06636194687296589
275. x = [0.7435523985838226, 0.5511456946987817], f([0.7435523985838226, 0.5511456946987817]) = 0.06606275358533634
276. x = [0.7435523999380909, 0.5528701737331867], f([0.7435523999380909, 0.5528701737331867]) = 0.06576537157751342
277. x = [0.7447041342605668, 0.5528701727094432], f([0.7447041342605668, 0.5528701727094432]) = 0.0654697843314729
278. x = [0.744704135625773, 0.554584251701755], f([0.744704135625773, 0.554584251701755]) = 0.06517597836658415
279. x = [0.7458471935096929, 0.5545842507688369], f([0.7458471935096929, 0.5545842507688369]) = 0.0648839374808796
280. x = [0.7458471946097084, 0.5562880394783292], f([0.7458471946097084, 0.5562880394783292]) = 0.06459364848775576
281. x = [0.7469816883577134, 0.5562880386877165], f([0.7469816883577134, 0.5562880386877165]) = 0.06430509549549375
282. x = [0.7469816890427691, 0.5579816444551484], f([0.7469816890427691, 0.5579816444551484]) = 0.06401826567965005
283. x = [0.7481077283371502, 0.5579816441480868], f([0.7481077283371502, 0.5579816441480868]) = 0.06373314352958406
284. x = [0.748107728867881, 0.5596651744224355], f([0.748107728867881, 0.5596651744224355]) = 0.06344971625609695
285. x = [0.7492254221917731, 0.5596651742313806], f([0.7492254221917731, 0.5596651742313806]) = 0.06316796885659984
286. x = [0.7492254220326316, 0.5613387328455446], f([0.7492254220326316, 0.5613387328455446]) = 0.0628878889547117
287. x = [0.7503348748294554, 0.5613387329227144], f([0.7503348748294554, 0.5613387329227144]) = 0.06260946165470319
288. x = [0.7503348746304412, 0.5630024239748729], f([0.7503348746304412, 0.5630024239748729]) = 0.06233267482579749
289. x = [0.751436190786186, 0.563002424024193], f([0.751436190786186, 0.563002424024193]) = 0.06205751397497599
290. x = [0.751436190976619, 0.5646563494836059], f([0.751436190976619, 0.5646563494836059]) = 0.06178396715621186
291. x = [0.7525294726866933, 0.5646563493190373], f([0.7525294726866933, 0.5646563493190373]) = 0.061512020307062716
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294. x = [0.7536148199853292, 0.5679352972509096], f([0.7536148199853292, 0.5679352972509096]) = 0.06070565693086175
295. x = [0.7546923304414226, 0.567935297098112], f([0.7546923304414226, 0.567935297098112]) = 0.060439985620871536
296. x = [0.7546923302375326, 0.5695605132261362], f([0.7546923302375326, 0.5695605132261362]) = 0.06017585284429178
297. x = [0.7557620999102656, 0.5695605132667891], f([0.7557620999102656, 0.5695605132667891]) = 0.05991324521179072

298. x = [0.7557621000756555, 0.5711763523059202], f([0.7557621000756555, 0.5711763523059202]) = 0.059652151759454146
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305. x = [0.7599656374185111, 0.5759685255016759], f([0.7599656374185111, 0.5759685255016759]) = 0.05786589655642971
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307. x = [0.76099808778028, 0.5775477709964268], f([0.76099808778028, 0.5775477709964268]) = 0.05736850409800208
308. x = [0.760998087985448, 0.5791180899826855], f([0.760998087985448, 0.5791180899826855]) = 0.05712191394661166
309. x = [0.7620233385781314, 0.5791180899547261], f([0.7620233385781314, 0.5791180899547261]) = 0.05687671291802648
310. x = [0.7620233389464247, 0.580679569562677], f([0.7620233389464247, 0.580679569562677]) = 0.05663289120620828
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312. x = [0.7630414758386912, 0.5822322949243053], f([0.7630414758386912, 0.5822322949243053]) = 0.05614934217270569
313. x = [0.764052580862826, 0.5822322944659357], f([0.764052580862826, 0.5822322944659357]) = 0.05590959421126847
314. x = [0.7640525814412245, 0.5837763479111554], f([0.7640525814412245, 0.5837763479111554]) = 0.055671184324550066
315. x = [0.7650567367932452, 0.5837763476114776], f([0.7650567367932452, 0.5837763476114776]) = 0.05543410155835036
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317. x = [0.7660540226507982, 0.5853118108281903], f([0.7660540226507982, 0.5853118108281903]) = 0.05496387941133433
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323. x = [0.7690054439459735, 0.589867468527183], f([0.7690054439459735, 0.589867468527183]) = 0.05358405657663759
324. x = [0.7690054443538296, 0.5913693738043562], f([0.7690054443538296, 0.5913693738043562]) = 0.053358484738171744
325. x = [0.7699760270624135, 0.5913693736537127], f([0.7699760270624135, 0.5913693736537127]) = 0.053134144663298306
326. x = [0.7699760271694124, 0.592863082331947], f([0.7699760271694124, 0.592863082331947]) = 0.0529110280767669l
327. x = [0.7709401212822132, 0.5928630823670048], f([0.7709401212822132, 0.5928630823670048]) = 0.05268912527879106
328. x = [0.7709401209202891, 0.5943486697889054], f([0.7709401209202891, 0.5943486697889054]) = 0.0524684282040118
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330. x = [0.7718977985531107, 0.5958262119621796], f([0.7718977985531107, 0.5958262119621796]) = 0.052030614304917296
331. x = [0.772849130970564, 0.5958262117323603], f([0.772849130970564, 0.5958262117323603]) = 0.051813480167353904
332. x = [0.7728491317154724, 0.5972957814691553], f([0.7728491317154724, 0.5972957814691553]) = 0.051597516962414915
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334. x = [0.7737941897959774, 0.5987574490288361], f([0.7737941897959774, 0.5987574490288361]) = 0.05116906857005839
335. x = [0.7747330407456483, 0.5987574486690677], f([0.7747330407456483, 0.5987574486690677]) = 0.05095656677164246
336. x = [0.7747330411494172, 0.6002112852100926], f([0.7747330411494172, 0.6002112852100926]) = 0.05074520274979016
337. x = [0.7756657533111521, 0.6002112851432372], f([0.7756657533111521, 0.6002112851432372]) = 0.0505349677352437
338. x = [0.7756657529580423, 0.6016573597986332], f([0.7756657529580423, 0.6016573597986332]) = 0.050325854395882136
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354. x = [0.7829142111891376, 0.6129546616752468], f([0.7829142111891376, 0.6129546616752468]) = 0.04712623970363435
355. x = [0.7837945426899662, 0.6129546618401578], f([0.7837945426899662, 0.6129546618401578]) = 0.04693502546464018
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357. x = [0.784669353572884, 0.6143338829937918], f([0.784669353572884, 0.6143338829937918]) = 0.04655555627183733
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364. x = [0.7872612303462622, 0.6197802430998092], f([0.7872612303462622, 0.6197802430998092]) = 0.04525778411378642
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434. x = [0.8143129788201156, 0.6631056282408125], f([0.8143129788201156, 0.6631056282408125]) = 0.03447966983465891
435. x = [0.8150095246988519, 0.6631056279535013], f([0.8150095246988519, 0.6631056279535013]) = 0.03435027516216825
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437. x = [0.8157022825010194, 0.6642405264426791], f([0.8157022825010194, 0.6642405264426791]) = 0.034093268000157065
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446. x = [0.8184360696267852, 0.6698376010130439], f([0.8184360696267852, 0.6698376010130439]) = 0.032965460812569694
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473. x = [0.8275590736447817, 0.6838121572991009], f([0.8275590736447817, 0.6838121572991009]) = 0.029844420948473123
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558. x = [0.8513668418386322, 0.7248255066707016], f([0.8513668418386322, 0.7248255066707016]) = 0.022091815705027485
559. x = [0.8518772749611748, 0.724825504167749], f([0.8518772749611748, 0.724825504167749]) = 0.022015925122841774
560. x = [0.8518772794263467, 0.7256949071015543], f([0.8518772794263467, 0.7256949071015543]) = 0.021940340350146807
561. x = [0.8523853546427204, 0.7256949043922417], f([0.8523853546427204, 0.7256949043922417]) = 0.021865059799051657

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569. x = [0.8543944174348697, 0.7291377252681362], f([0.8543944174348697, 0.7291377252681362]) = 0.021273592310023817
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579. x = [0.8568545771060888, 0.7333644728729364], f([0.8568545771060888, 0.7333644728729364]) = 0.020560383607684952
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583. x = [0.8578231083360324, 0.7350317803147687], f([0.8578231083360324, 0.7350317803147687]) = 0.020282943701127962
584. x = [0.8578231128597504, 0.7358605010327147], f([0.8578231128597504, 0.7358605010327147]) = 0.020214267236897784
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611. x = [0.8643654746885397, 0.746343085486948], f([0.8643654746885397, 0.746343085486948]) = 0.01845828234382666
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707. x = [0.8840300726936869, 0.780853255457645], f([0.8840300726936869, 0.780853255457645]) = 0.013492046352925514
708. x = [0.8840300766191581, 0.7815091833619677], f([0.8840300766191581, 0.7815091833619677]) = 0.01344902312896323
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834. x = [0.9047097434328836, 0.8184997295225044], f([0.9047097434328836, 0.8184997295225044]) = 0.009080232996636206
835. x = [0.9049997716778948, 0.8184997265823786], f([0.9049997716778948, 0.8184997265823786]) = 0.009052591199447419
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837. x = [0.9052887389208047, 0.8190246043170042], f([0.9052887389208047, 0.8190246043170042]) = 0.00899758597002354
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839. x = [0.9055766447363368, 0.8195477186531898], f([0.9055766447363368, 0.8195477186531898]) = 0.008942949646260906
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961. x = [0.9213181864532358, 0.848400198392589], f([0.9213181864532358, 0.848400198392589]) = 0.006209060879162732
962. x = [0.9213181946257178, 0.8488272309551849], f([0.9213181946257178, 0.8488272309551849]) = 0.006190826496979541
963. x = [0.9215491677075497, 0.8488272264913388], f([0.9215491677075497, 0.8488272264913388]) = 0.006172650199542785
964. x = [0.9215491759488464, 0.8492528987020144], f([0.9215491759488464, 0.8492528987020144]) = 0.006154531794327583
965. x = [0.9217793560787022, 0.8492528942981273], f([0.9217793560787022, 0.8492528942981273]) = 0.006136471080852969
966. x = [0.9217793645329273, 0.8496772123692661], f([0.9217793645329273, 0.8496772123692661]) = 0.0061184678128966665
967. x = [0.9220087554719467, 0.8496772078266547], f([0.9220087554719467, 0.8496772078266547]) = 0.006100521822414349
968. x = [0.9220087640563741, 0.8501001768411702], f([0.9220087640563741, 0.8501001768411702]) = 0.006082632884039428
969. x = [0.9222373692451884, 0.8501001721970868], f([0.9222373692451884, 0.8501001721970868]) = 0.006064800810642016

970. x = [0.9222373778500381, 0.8505217970781929], f([0.9222373778500381, 0.8505217970781929]) = 0.006047025403663261
971. x = [0.9224652006801186, 0.8505217923982927], f([0.9224652006801186, 0.8505217923982927]) = 0.006029306453700887
972. x = [0.9224652091684857, 0.8509420776078945], f([0.9224652091684857, 0.8509420776078945]) = 0.0060116437893106365
973. x = [0.9226922528370642, 0.8509420730746031], f([0.9226922528370642, 0.8509420730746031]) = 0.005994037199126726
974. x = [0.9226922611476706, 0.8513610241586341], f([0.9226922611476706, 0.8513610241586341]) = 0.005976486486483612
975. x = [0.9229185292187964, 0.8513610196582326], f([0.9229185292187964, 0.8513610196582326]) = 0.0059589914387207545
976. x = [0.9229185372911985, 0.8517786413888238], f([0.9229185372911985, 0.8517786413888238]) = 0.005941551893350589
977. x = [0.9231440329260162, 0.8517786370262723], f([0.9231440329260162, 0.8517786370262723]) = 0.0059241676213395685
978. x = [0.9231440410324243, 0.8521949355739284], f([0.9231440410324243, 0.8521949355739284]) = 0.005906838428848419
979. x = [0.9233687677902817, 0.8521949311646232], f([0.9233687677902817, 0.8521949311646232]) = 0.005889564113999466
980. x = [0.923368776066187, 0.8526099118002777], f([0.923368776066187, 0.8526099118002777]) = 0.005872344481617265
981. x = [0.9235927374346581, 0.8526099073621101], f([0.9235927374346581, 0.8526099073621101]) = 0.005855179352664315
982. x = [0.9235927452854975, 0.8530235734442081], f([0.9235927452854975, 0.8530235734442081]) = 0.005838068573027313
983. x = [0.9238159440873416, 0.8530235692670151], f([0.9238159440873416, 0.8530235692670151]) = 0.005821011919062619
984. x = [0.9238159519153692, 0.8534359273558936], f([0.9238159519153692, 0.8534359273558936]) = 0.005804009182581902
985. x = [0.9240383917379903, 0.8534359231683332], f([0.9240383917379903, 0.8534359231683332]) = 0.005787060186534011
986. x = [0.9240383997744553, 0.8538469790924065], f([0.9240383997744553, 0.8538469790924065]) = 0.005770164708847472
987. x = [0.9242600841462311, 0.8538469747632509], f([0.9242600841462311, 0.8538469747632509]) = 0.005753322588298864
988. x = [0.9242600921263099, 0.8542567324742576], f([0.9242600921263099, 0.8542567324742576]) = 0.005736533644736315
989. x = [0.9244810240911069, 0.8542567282223464], f([0.9244810240911069, 0.8542567282223464]) = 0.005719797692977011
990. x = [0.9244810317752236, 0.8546651920875198], f([0.9244810317752236, 0.8546651920875198]) = 0.005703114561754325
991. x = [0.9247012143802424, 0.8546651880130189], f([0.9247012143802424, 0.8546651880130189]) = 0.005686484054067238
992. x = [0.9247012219897596, 0.8550723640750929], f([0.9247012219897596, 0.8550723640750929]) = 0.005669905969855421
993. x = [0.9249206586581258, 0.8550723599586949], f([0.9249206586581258, 0.8550723599586949]) = 0.005653380124291675

```
994. x = [0.9249206664961406, 0.8554782538821156], f([0.9249206664961406,
0.8554782538821156]) = 0.005636906319404978
995. x = [0.9251393605452825, 0.8554782496381296], f([0.9251393605452825,
0.8554782496381296]) = 0.0056204843867956635
996. x = [0.9251393686083441, 0.8558828662716627], f([0.9251393686083441,
0.8558828662716627]) = 0.005604114132379643
997. x = [0.9253573233284182, 0.855882861927143], f([0.9253573233284182,
0.855882861927143]) = 0.005587795391729842
998. x = [0.9253573311345317, 0.8562862047707357], f([0.9253573311345317,
0.8562862047707357]) = 0.0055715280153809405
999. x = [0.9255745493698055, 0.8562862005552137], f([0.9255745493698055,
0.8562862005552137]) = 0.005555311790944824
1000. x = [0.9255745569972949, 0.8566882745000424], f([0.9255745569972949,
0.8566882745000424]) = 0.005539146566168343
WARNING: maximum number of iterations reached
```

[26]:
```
([0.9255745569972949, 0.8566882745000424], [1.0 0.0 0.0; 0.7711096853441531
 0.1612620227407261 0.0; … ; 0.005555311790944824 0.9255745493698055
 0.8562862005552137; 0.005539146566168343 0.9255745569972949 0.8566882745000424])
```

The minimizer is located at $(1, 1)$. Indeed,

$$\nabla f(1,1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\nabla^2 f(1,1) = \begin{pmatrix} 802 & -400 \\ -400 & 200 \end{pmatrix}$$

The determinants of the principal minors are positive as they are respectively 802 and $802 \times 200 - 400^2 = 400$, so the Hessian is positive definite.

However the steepest descent method converges very slowly.

[ ]:
```
plot!(iter[:,2], iter[:,3])
```

2 Exact minimization of approximate minimization?

The exact minimization of the function along the search direction requires assumptions as unimodality or convexity, that are not necessarily satisfied. It is more practical to approximately minimize the function along the search direction using backtracking. This will be done more explicitely in the linesearch notebook.

For nonconvex functions, a first approach is to fix the step length.

[ ]:
```
function batchdescent(f::Function, fprime::Function, x0, ::Float64, verbose::
↪Bool = true,
                        record::Bool = false, tol::Float64 = 1e-7, maxiter::Int64␣
↪= 1000)
```

```
    function fsearch( ::Float64)
        return(f(x- *grad))
    end

    x = x0
    k = 0

    grad = fprime(x)

    if (verbose || record)
        fx = f(x)
    end
    if (verbose)
        println("$k. x = $x, f($x) = $fx")
    end
    if (record)
        iterates = [ fx x' ]
    end

    while ((k < maxiter) && (norm(grad) > tol))
        x = x- *grad
        k += 1
        grad = fprime(x)
        if (verbose || record)
            fx = f(x)
        end
        if (verbose)
            println("$k. x = $x, f($x) = $fx")
        end
        if (record)
            iterates = [ iterates; fx x' ]
        end
    end

    if (k == maxiter)
        println("WARNING: maximum number of iterations reached")
    end

    if (record)
        return x, iterates
    else
        return x
    end
end
```

We can get close too the solution if α is small enough.

```
[ ]: sol, iter = batchdescent(f1, f1grad, [2.0,3.0], 0.1, true, true)
```

But if α is too large, it does not work at all!

```
[ ]: ol, iter = batchdescent(f1, f1grad, [2.0,3.0], 2.0, true, true)
```

If $f \in C^1$, f convex, and $\nabla f(\cdot)$ is Lipschitz continuous, i.e. $\exists L > 0$ such that

$$\forall x, y, \quad \|\nabla f(x) - \nabla f(y)\|_2 \leq L\|x - y\|_2,$$

we can recover the convergence by considering a decreasing sequence of step lengths $\alpha_k > 0$ staisfying

$$\sum_{k=1}^{+\infty} \alpha_k = +\infty, \qquad \sum_{k=1}^{+\infty} \alpha_k^2 < +\infty.$$

Example: $\alpha_k = \frac{\kappa}{k}$.

```
[ ]: function rbdescent(f::Function, fprime::Function, x0,  0::Float64, verbose::
     ↪Bool = true,
                     record::Bool = false, tol::Float64 = 1e-7, maxiter::Int64 =␣
     ↪1000)

         function fsearch( ::Float64)
             return(f(x- *grad))
         end

         x = x0
         k = 0
           =  0

         grad = fprime(x)

         if (verbose || record)
             fx = f(x)
         end
         if (verbose)
             println("$k. x = $x, f($x) = $fx")
         end
         if (record)
             iterates = [ fx x' ]
         end

         while ((k < maxiter) && (norm(grad) > tol))
             k += 1
               =  0/k
             x = x- *grad
             grad = fprime(x)
             if (verbose || record)
```

```
            fx = f(x)
        end
        if (verbose)
            println("$k. x = $x, f($x) = $fx", ",   = ",  )
        end
        if (record)
            iterates = [ iterates; fx x' ]
        end
    end

    if (k == maxiter)
        println("WARNING: maximum number of iterations reached")
    end

    if (record)
        return x, iterates
    else
        return x
    end
end
```

[ ]:
```
ol, iter = rbdescent(f1, f1grad, [2.0,3.0], 2.0, true, true)
```

[ ]:
```
ol, iter = rbdescent(f1, f1grad, [10.0,10.0], 2.0, true, true)
```

[ ]:
```
ol, iter = rbdescent(f1, f1grad, [100.0,100.0], 2.0, true, true)
```

[ ]:
```
ol, iter = rbdescent(f1, f1grad, [100.0,100.0], 0.1, true, true)
```

This technique has been proposed by Robbins and Monro in 1951 in the context of stochastic approximation, where the objective is

$$f(x) = E[g(x, \xi)]$$

and at each iteration, the next iterate is computed as

$$x_{k+1} = x_k - \alpha_k \nabla g(x_k, \xi_k)$$

where ξ_k is drawn from the distribution of ξ.

This technique, as well as some extensions (mini-batch, stochastic average gradient,…) is still very popular in machine learning.