

linesearch

June 16, 2020

1 Linesearch methods

Consider $f \in C^2$. A descent method consists to iteratively compute

$$x_{k+1} = x_k + \alpha^* d_k$$

where α^* approximately minimizes $f(x_k - \alpha d_k)$.

```
[1]: using Optim
      using Plots
      plotly()
```

```
Info: Precompiling Optim [429524aa-4258-5aef-a3af-852621145aeb]
@ Base loading.jl:1273
WARNING: Method definition vec(Number) in module FiniteDiff at
C:\Users\slash\.julia\packages\FiniteDiff\EBys0\src\jacobians.jl:128 overwritten
in module DiffEqDiffTools at
C:\Users\slash\.julia\packages\DiffEqDiffTools\3mm8U\src\jacobians.jl:114.
** incremental compilation may be fatally broken for this module **

Info: For saving to png with the Plotly backend ORCA has to be installed.
@ Plots C:\Users\slash\.julia\packages\Plots\qZHsp\src\backends.jl:363
```

```
[1]: Plots.PlotlyBackend()
```

Several linesearch techniques are proposed in Julia, as explained in the page <https://github.com/JuliaNLSolvers/LineSearches.jl>

Consider again the Rosenbrock example, whose mathematical expression is

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2$$

Its gradient can be computed as

$$\nabla f(x, y) = \begin{pmatrix} -2(1 - x) - 400x(y - x^2) \\ 200(y - x^2) \end{pmatrix}$$
$$\nabla^2 f(x, y) = \begin{pmatrix} 2 - 400(y - x^2) + 800x^2 & -400x \\ -400x & 200 \end{pmatrix} = \begin{pmatrix} 2 - 400y + 1200x^2 & -400x \\ -400x & 200 \end{pmatrix}$$

The minimizer is located at (1, 1). Indeed,

$$\nabla f(1, 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and

$$\nabla^2 f(1, 1) = \begin{pmatrix} 802 & -400 \\ -400 & 200 \end{pmatrix}$$

The determinants of the principal minors are positive as they are respectively 802 and $802 \times 200 - 400^2 = 400$, so the Hessian is positive definite.

```
[3]: # Rosenbrock function
# Source: https://bitbucket.org/lurk3r/optim.jl

function rosenbrock(x::Vector)
    return (1.0 - x[1])^2 + 100.0 * (x[2] - x[1]^2)^2
end

function rosenbrock_gradient!(storage::Vector, x::Vector)
    storage[1] = -2.0 * (1.0 - x[1]) - 400.0 * (x[2] - x[1]^2) * x[1]
    storage[2] = 200.0 * (x[2] - x[1]^2)
end

function rosenbrock_hessian!(storage::Matrix, x::Vector)
    storage[1, 1] = 2.0 - 400.0 * x[2] + 1200.0 * x[1]^2
    storage[1, 2] = -400.0 * x[1]
    storage[2, 1] = -400.0 * x[1]
    storage[2, 2] = 200.0
end
```

```
[3]: rosenbrock_hessian! (generic function with 1 method)
```

```
[4]: using Plots

default(size=(600,600), fc=:heat)
x, y = 0:0.02:1.0, 0:0.02:1.0
z = Surface((x,y)->rosenbrock([x,y]), x, y)
surface(x,y,z, linealpha = 0.3)
```

```
[5]: Plots.contour(x,y,z, linealpha = 0.1, levels=1600)
```

We can optimize it using the optimize function present in the package optim.jl

```
[8]: res = optimize(rosenbrock, rosenbrock_gradient!,
    [20.0, 20.0],
    Optim.GradientDescent(),
    Optim.Options(g_tol = 1e-6,
        store_trace = true,
        show_trace = true))
```

Iter	Function value	Gradient norm
0	1.444036e+07	3.040038e+06
* time: 0.0		
1	2.680958e+06	8.881503e+05
* time: 0.003000020980834961		
2	6.986401e+02	4.430920e+03
* time: 0.006999969482421875		
3	1.422712e+01	2.423975e+02
* time: 0.020999908447265625		
4	1.241984e+01	7.750435e-01
* time: 0.03600001335144043		
5	1.241547e+01	1.200614e+01
* time: 0.0409998893737793		
6	1.241110e+01	8.544267e-01
* time: 0.04399991035461426		
7	1.241103e+01	8.557174e-01
* time: 0.056999921798706055		
8	1.241096e+01	8.544292e-01
* time: 0.06999993324279785		
9	1.241089e+01	8.557152e-01
* time: 0.09500002861022949		
10	1.241081e+01	8.544285e-01
* time: 0.12899994850158691		
11	1.241074e+01	8.557130e-01
* time: 0.16000008583068848		
12	1.241067e+01	8.544278e-01
* time: 0.18799996376037598		
13	1.241060e+01	8.557108e-01
* time: 0.20300006866455078		
14	1.241053e+01	8.544272e-01
* time: 0.22799992561340332		
15	1.241045e+01	8.557086e-01
* time: 0.24699997901916504		
16	1.241038e+01	8.544265e-01
* time: 0.2609999179840088		
17	1.241031e+01	8.557064e-01
* time: 0.26399993896484375		
18	1.241024e+01	8.544259e-01
* time: 0.2779998779296875		
19	1.241016e+01	8.557042e-01
* time: 0.29600000381469727		
20	1.241009e+01	8.544252e-01
* time: 0.309999942779541		
21	1.241002e+01	8.557020e-01
* time: 0.32399988174438477		
22	1.240995e+01	8.544246e-01
* time: 0.3280000686645508		
23	1.240987e+01	8.556998e-01

```

* time: 0.34599995613098145
  24    1.240980e+01    8.544239e-01
* time: 0.3510000705718994
  25    1.240973e+01    8.556976e-01
* time: 0.36700010299682617
  26    1.240966e+01    8.544232e-01
* time: 0.3689999580383301
  27    1.240958e+01    8.556955e-01
* time: 0.38299989700317383
  28    1.240951e+01    8.544226e-01
* time: 0.38700008392333984
  29    1.240944e+01    8.556933e-01
* time: 0.40199995040893555
  30    1.240937e+01    8.544219e-01
* time: 0.4049999713897705
  31    1.240930e+01    8.556911e-01
* time: 0.4089999198913574
  32    1.240922e+01    8.544213e-01
* time: 0.4140000343322754
  33    1.240915e+01    8.556889e-01
* time: 0.42100000381469727
  34    1.240908e+01    8.544206e-01
* time: 0.4419999122619629
  35    1.240901e+01    8.556867e-01
* time: 0.4679999351501465
  36    1.240893e+01    8.544199e-01
* time: 0.4830000400543213
  37    1.240886e+01    8.556845e-01
* time: 0.4960000514984131
  38    1.240879e+01    8.544193e-01
* time: 0.5090000629425049
  39    1.240872e+01    8.556823e-01
* time: 0.5130000114440918
  40    1.240864e+01    8.544186e-01
* time: 0.5269999504089355
  41    1.240857e+01    8.556802e-01
* time: 0.5499999523162842
  42    1.240850e+01    8.544180e-01
* time: 0.5539999008178711
  43    1.240843e+01    8.556780e-01
* time: 0.5690000057220459
  44    1.240835e+01    8.544173e-01
* time: 0.5820000171661377
  45    1.240828e+01    8.556758e-01
* time: 0.5980000495910645
  46    1.240821e+01    8.544167e-01
* time: 0.6150000095367432
  47    1.240814e+01    8.556736e-01

```

```

* time: 0.6189999580383301
  48    1.240806e+01    8.544160e-01
* time: 0.6329998970031738
  49    1.240799e+01    8.556714e-01
* time: 0.6470000743865967
  50    1.240792e+01    8.544153e-01
* time: 0.6730000972747803
  51    1.240785e+01    8.556693e-01
* time: 0.6779999732971191
  52    1.240778e+01    8.544147e-01
* time: 0.6909999847412109
  53    1.240770e+01    8.556671e-01
* time: 0.7070000171661377
  54    1.240763e+01    8.544140e-01
* time: 0.7109999656677246
  55    1.240756e+01    8.556649e-01
* time: 0.7130000591278076
  56    1.240749e+01    8.544134e-01
* time: 0.7260000705718994
  57    1.240741e+01    8.556627e-01
* time: 0.7330000400543213
  58    1.240734e+01    8.544127e-01
* time: 0.747999906539917
  59    1.240727e+01    8.556605e-01
* time: 0.75
  60    1.240720e+01    8.544120e-01
* time: 0.7630000114440918
  61    1.240712e+01    8.556584e-01
* time: 0.7909998893737793
  62    1.240705e+01    8.544114e-01
* time: 0.8169999122619629
  63    1.240698e+01    8.556562e-01
* time: 0.8310000896453857
  64    1.240691e+01    8.544107e-01
* time: 0.8340001106262207
  65    1.240683e+01    8.556540e-01
* time: 0.8469998836517334
  66    1.240676e+01    8.544101e-01
* time: 0.8910000324249268
  67    1.240669e+01    8.556518e-01
* time: 0.9049999713897705
  68    1.240662e+01    8.544094e-01
* time: 0.9110000133514404
  69    1.240654e+01    8.556497e-01
* time: 0.9259998798370361
  70    1.240647e+01    8.544087e-01
* time: 0.9289999008178711
  71    1.240640e+01    8.556475e-01

```

```

* time: 0.9309999942779541
  72    1.240633e+01    8.544081e-01
* time: 0.9600000381469727
  73    1.240626e+01    8.556453e-01
* time: 0.9760000705718994
  74    1.240618e+01    8.544074e-01
* time: 1.0079998970031738
  75    1.240611e+01    8.556432e-01
* time: 1.0409998893737793
  76    1.240604e+01    8.544068e-01
* time: 1.059000015258789
  77    1.240597e+01    8.556410e-01
* time: 1.065999984741211
  78    1.240589e+01    8.544061e-01
* time: 1.0799999237060547
  79    1.240582e+01    8.556388e-01
* time: 1.1050000190734863
  80    1.240575e+01    8.544054e-01
* time: 1.1319999694824219
  81    1.240568e+01    8.556366e-01
* time: 1.1510000228881836
  82    1.240560e+01    8.544048e-01
* time: 1.1589999198913574
  83    1.240553e+01    8.556345e-01
* time: 1.1630001068115234
  84    1.240546e+01    8.544041e-01
* time: 1.1989998817443848
  85    1.240539e+01    8.556323e-01
* time: 1.2009999752044678
  86    1.240531e+01    8.544035e-01
* time: 1.2279999256134033
  87    1.240524e+01    8.556301e-01
* time: 1.250999927520752
  88    1.240517e+01    8.544028e-01
* time: 1.253000020980835
  89    1.240510e+01    8.556280e-01
* time: 1.2669999599456787
  90    1.240502e+01    8.544021e-01
* time: 1.2809998989105225
  91    1.240495e+01    8.556258e-01
* time: 1.2999999523162842
  92    1.240488e+01    8.544015e-01
* time: 1.3269999027252197
  93    1.240481e+01    8.556236e-01
* time: 1.3480000495910645
  94    1.240474e+01    8.544008e-01
* time: 1.3650000095367432
  95    1.240466e+01    8.556215e-01

```

```

* time: 1.371999979019165
  96      1.240459e+01      8.544002e-01
* time: 1.378999948501587
  97      1.240452e+01      8.556193e-01
* time: 1.4010000228881836
  98      1.240445e+01      8.543995e-01
* time: 1.4219999313354492
  99      1.240437e+01      8.556172e-01
* time: 1.4259998798370361
 100      1.240430e+01      8.543988e-01
* time: 1.4300000667572021
 101      1.240423e+01      8.556150e-01
* time: 1.4470000267028809
 102      1.240416e+01      8.543982e-01
* time: 1.4830000400543213
 103      1.240408e+01      8.556128e-01
* time: 1.5090000629425049
 104      1.240401e+01      8.543975e-01
* time: 1.5269999504089355
 105      1.240394e+01      8.556107e-01
* time: 1.5469999313354492
 106      1.240387e+01      8.543968e-01
* time: 1.561000108718872
 107      1.240379e+01      8.556085e-01
* time: 1.5899999141693115
 108      1.240372e+01      8.543962e-01
* time: 1.5950000286102295
 109      1.240365e+01      8.556063e-01
* time: 1.6129999160766602
 110      1.240358e+01      8.543955e-01
* time: 1.6299998760223389
 111      1.240350e+01      8.556042e-01
* time: 1.6449999809265137
 112      1.240343e+01      8.543949e-01
* time: 1.6500000953674316
 113      1.240336e+01      8.556020e-01
* time: 1.6670000553131104
 114      1.240329e+01      8.543942e-01
* time: 1.6800000667572021
 115      1.240322e+01      8.555999e-01
* time: 1.696000099182129
 116      1.240314e+01      8.543935e-01
* time: 1.7119998931884766
 117      1.240307e+01      8.555977e-01
* time: 1.7279999256134033
 118      1.240300e+01      8.543929e-01
* time: 1.7320001125335693
 119      1.240293e+01      8.555956e-01

```

```

* time: 1.7609999179840088
  120    1.240285e+01    8.543922e-01
* time: 1.7769999504089355
  121    1.240278e+01    8.555934e-01
* time: 1.7969999313354492
  122    1.240271e+01    8.543915e-01
* time: 1.8029999732971191
  123    1.240264e+01    8.555912e-01
* time: 1.816999912261963
  124    1.240256e+01    8.543909e-01
* time: 1.8299999237060547
  125    1.240249e+01    8.555891e-01
* time: 1.8589999675750732
  126    1.240242e+01    8.543902e-01
* time: 1.877000093460083
  127    1.240235e+01    8.555869e-01
* time: 1.8970000743865967
  128    1.240227e+01    8.543896e-01
* time: 1.9259998798370361
  129    1.240220e+01    8.555848e-01
* time: 1.9300000667572021
  130    1.240213e+01    8.543889e-01
* time: 1.9719998836517334
  131    1.240206e+01    8.555826e-01
* time: 2.007999897003174
  132    1.240198e+01    8.543882e-01
* time: 2.0230000019073486
  133    1.240191e+01    8.555805e-01
* time: 2.0429999828338623
  134    1.240184e+01    8.543876e-01
* time: 2.049999952316284
  135    1.240177e+01    8.555783e-01
* time: 2.0859999656677246
  136    1.240169e+01    8.543869e-01
* time: 2.131999969482422
  137    1.240162e+01    8.555762e-01
* time: 2.1640000343322754
  138    1.240155e+01    8.543862e-01
* time: 2.1710000038146973
  139    1.240148e+01    8.555740e-01
* time: 2.194999933242798
  140    1.240141e+01    8.543856e-01
* time: 2.2130000591278076
  141    1.240133e+01    8.555719e-01
* time: 2.239000082015991
  142    1.240126e+01    8.543849e-01
* time: 2.246999979019165
  143    1.240119e+01    8.555697e-01

```



```

* time: 2.265000104904175
  144    1.240112e+01    8.543843e-01
* time: 2.2709999084472656
  145    1.240104e+01    8.555676e-01
* time: 2.2769999504089355
  146    1.240097e+01    8.543836e-01
* time: 2.305999994277954
  147    1.240090e+01    8.555654e-01
* time: 2.3269999027252197
  148    1.240083e+01    8.543829e-01
* time: 2.3299999237060547
  149    1.240075e+01    8.555633e-01
* time: 2.3450000286102295
  150    1.240068e+01    8.543823e-01
* time: 2.359999895095825
  151    1.240061e+01    8.555611e-01
* time: 2.371999979019165
  152    1.240054e+01    8.543816e-01
* time: 2.3889999389648438
  153    1.240046e+01    8.555590e-01
* time: 2.3959999084472656
  154    1.240039e+01    8.543809e-01
* time: 2.4119999408721924
  155    1.240032e+01    8.555569e-01
* time: 2.430000066757202
  156    1.240025e+01    8.543803e-01
* time: 2.450000047683716
  157    1.240017e+01    8.555547e-01
* time: 2.4630000591278076
  158    1.240010e+01    8.543796e-01
* time: 2.4839999675750732
  159    1.240003e+01    8.555526e-01
* time: 2.490000009536743
  160    1.239996e+01    8.543789e-01
* time: 2.50600004196167
  161    1.239988e+01    8.555504e-01
* time: 2.5239999294281006
  162    1.239981e+01    8.543783e-01
* time: 2.555000066757202
  163    1.239974e+01    8.555483e-01
* time: 2.5759999752044678
  164    1.239967e+01    8.543776e-01
* time: 2.5940001010894775
  165    1.239960e+01    8.555461e-01
* time: 2.6019999980926514
  166    1.239952e+01    8.543769e-01
* time: 2.6110000610351562
  167    1.239945e+01    8.555440e-01

```

```

* time: 2.615000009536743
  168    1.239938e+01    8.543763e-01
* time: 2.6529998779296875
  169    1.239931e+01    8.555419e-01
* time: 2.686000108718872
  170    1.239923e+01    8.543756e-01
* time: 2.7330000400543213
  171    1.239916e+01    8.555397e-01
* time: 2.736999988555908
  172    1.239909e+01    8.543750e-01
* time: 2.739000082015991
  173    1.239902e+01    8.555376e-01
* time: 2.754999876022339
  174    1.239894e+01    8.543743e-01
* time: 2.7709999084472656
  175    1.239887e+01    8.555354e-01
* time: 2.7920000553131104
  176    1.239880e+01    8.543736e-01
* time: 2.809000015258789
  177    1.239873e+01    8.555333e-01
* time: 2.822000026702881
  178    1.239865e+01    8.543730e-01
* time: 2.8310000896453857
  179    1.239858e+01    8.555312e-01
* time: 2.8459999561309814
  180    1.239851e+01    8.543723e-01
* time: 2.8540000915527344
  181    1.239844e+01    8.555290e-01
* time: 2.871000051498413
  182    1.239836e+01    8.543716e-01
* time: 2.874000072479248
  183    1.239829e+01    8.555269e-01
* time: 2.9010000228881836
  184    1.239822e+01    8.543710e-01
* time: 2.9159998893737793
  185    1.239815e+01    8.555248e-01
* time: 2.934000015258789
  186    1.239807e+01    8.543703e-01
* time: 2.937999963760376
  187    1.239800e+01    8.555226e-01
* time: 2.944999933242798
  188    1.239793e+01    8.543696e-01
* time: 2.9649999141693115
  189    1.239786e+01    8.555205e-01
* time: 2.99399995803833
  190    1.239779e+01    8.543690e-01
* time: 2.999000072479248
  191    1.239771e+01    8.555184e-01

```

```

* time: 3.015000104904175
  192    1.239764e+01    8.543683e-01
* time: 3.0309998989105225
  193    1.239757e+01    8.555162e-01
* time: 3.0460000038146973
  194    1.239750e+01    8.543676e-01
* time: 3.062000036239624
  195    1.239742e+01    8.555141e-01
* time: 3.0869998931884766
  196    1.239735e+01    8.543670e-01
* time: 3.11199988555908
  197    1.239728e+01    8.555120e-01
* time: 3.115000009536743
  198    1.239721e+01    8.543663e-01
* time: 3.128000020980835
  199    1.239713e+01    8.555099e-01
* time: 3.1459999084472656
  200    1.239706e+01    8.543656e-01
* time: 3.1649999618530273
  201    1.239699e+01    8.555077e-01
* time: 3.190000057220459
  202    1.239692e+01    8.543650e-01
* time: 3.194999933242798
  203    1.239684e+01    8.555056e-01
* time: 3.2139999866485596
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  205    1.239670e+01    8.555035e-01
* time: 3.2239999771118164
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  207    1.239655e+01    8.555013e-01
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* time: 3.2360000610351562
  209    1.239641e+01    8.554992e-01
* time: 3.239000082015991
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* time: 3.2639999389648438
  211    1.239626e+01    8.554971e-01
* time: 3.2790000438690186
  212    1.239619e+01    8.543616e-01
* time: 3.2939999103546143
  213    1.239612e+01    8.554950e-01
* time: 3.296999931335449
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* time: 3.322999954223633
  215    1.239597e+01    8.554928e-01

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* time: 3.3510000705718994
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* time: 3.384000062942505
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* time: 3.3989999294281006
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* time: 3.4059998989105225
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* time: 3.4509999752044678
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  223    1.239540e+01    8.554843e-01
* time: 3.4570000171661377
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* time: 3.4830000400543213
  225    1.239525e+01    8.554822e-01
* time: 3.500999927520752
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* time: 3.504999876022339
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* time: 3.510999917984009
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* time: 3.5429999828338623
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* time: 3.5460000038146973
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* time: 3.552000045776367
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* time: 3.555000066757202
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* time: 3.5789999961853027
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* time: 3.5950000286102295
  235    1.239453e+01    8.554716e-01
* time: 3.624000072479248
  236    1.239445e+01    8.543536e-01
* time: 3.6459999084472656
  237    1.239438e+01    8.554695e-01
* time: 3.683000087738037
  238    1.239431e+01    8.543530e-01
* time: 3.697000026702881
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* time: 3.9560000896453857
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* time: 4.016999959945679
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* time: 4.049999952316284
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* time: 4.07699990272522
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* time: 4.0899999141693115
  259    1.239279e+01    8.554463e-01
* time: 4.107000112533569
  260    1.239272e+01    8.543456e-01
* time: 4.127000093460083
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* time: 4.144000053405762
  262    1.239257e+01    8.543449e-01
* time: 4.1519999504089355
  263    1.239250e+01    8.554421e-01

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* time: 4.204999923706055
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* time: 4.33299994468689
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* time: 4.4019999504089355
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* time: 4.42300009727478
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* time: 4.427000045776367
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* time: 4.444999933242798
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* time: 4.447999954223633
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* time: 4.476000070571899
  287    1.239076e+01    8.554168e-01

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* time: 4.629999876022339
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* time: 4.7779998779296875
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* time: 4.793999910354614
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* time: 4.811000108718872
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* time: 4.83299994468689
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* time: 4.854000091552734
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* time: 4.857000112533569
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* time: 4.882999897003174
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* time: 4.914999961853027
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* time: 4.974999904632568
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* time: 4.99399995803833
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* time: 5.585000038146973
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* time: 6.575000047683716
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* time: 6.628000020980835
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* time: 7.549999952316284
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* time: 7.624000072479248
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* time: 8.76799988746643
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* time: 9.036999940872192
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* time: 9.05299997329712
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* time: 9.109999895095825
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* time: 9.170000076293945
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* time: 9.782000064849854
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* time: 9.928999900817871
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* time: 18.109999895095825
  919    1.234494e+01    8.547839e-01
* time: 18.115999937057495
  920    1.234487e+01    8.541201e-01
* time: 18.121000051498413
  921    1.234479e+01    8.547820e-01
* time: 18.128999948501587
  922    1.234472e+01    8.541194e-01
* time: 18.145999908447266
  923    1.234465e+01    8.547801e-01
* time: 18.161999940872192
  924    1.234458e+01    8.541187e-01
* time: 18.17899990081787
  925    1.234450e+01    8.547782e-01
* time: 18.197999954223633
  926    1.234443e+01    8.541181e-01
* time: 18.216000080108643
  927    1.234436e+01    8.547762e-01
* time: 18.223000049591064
  928    1.234429e+01    8.541174e-01
* time: 18.23900008201599
  929    1.234421e+01    8.547743e-01
* time: 18.26200008392334
  930    1.234414e+01    8.541167e-01
* time: 18.27900004386902
  931    1.234407e+01    8.547724e-01
* time: 18.296000003814697
  932    1.234400e+01    8.541160e-01
* time: 18.299999952316284
  933    1.234392e+01    8.547705e-01
* time: 18.332000017166138
  934    1.234385e+01    8.541153e-01
* time: 18.346999883651733
  935    1.234378e+01    8.547686e-01

```

```

* time: 18.365999937057495
  936    1.234371e+01    8.541146e-01
* time: 18.38100004196167
  937    1.234363e+01    8.547667e-01
* time: 18.405999898910522
  938    1.234356e+01    8.541139e-01
* time: 18.41100001335144
  939    1.234349e+01    8.547648e-01
* time: 18.430000066757202
  940    1.234342e+01    8.541132e-01
* time: 18.44700002670288
  941    1.234334e+01    8.547629e-01
* time: 18.46499991416931
  942    1.234327e+01    8.541125e-01
* time: 18.483999967575073
  943    1.234320e+01    8.547610e-01
* time: 18.520999908447266
  944    1.234312e+01    8.541118e-01
* time: 18.543999910354614
  945    1.234305e+01    8.547591e-01
* time: 18.563999891281128
  946    1.234298e+01    8.541111e-01
* time: 18.6010000705719
  947    1.234291e+01    8.547572e-01
* time: 18.63700008392334
  948    1.234283e+01    8.541104e-01
* time: 18.67199993133545
  949    1.234276e+01    8.547553e-01
* time: 18.688999891281128
  950    1.234269e+01    8.541097e-01
* time: 18.694999933242798
  951    1.234262e+01    8.547534e-01
* time: 18.698999881744385
  952    1.234254e+01    8.541090e-01
* time: 18.717000007629395
  953    1.234247e+01    8.547515e-01
* time: 18.73300004005432
  954    1.234240e+01    8.541083e-01
* time: 18.752000093460083
  955    1.234233e+01    8.547496e-01
* time: 18.77500009536743
  956    1.234225e+01    8.541076e-01
* time: 18.795000076293945
  957    1.234218e+01    8.547477e-01
* time: 18.812999963760376
  958    1.234211e+01    8.541069e-01
* time: 18.841000080108643
  959    1.234204e+01    8.547458e-01

```

```

* time: 18.84999990463257
  960    1.234196e+01    8.541062e-01
* time: 18.86400008201599
  961    1.234189e+01    8.547439e-01
* time: 18.871999979019165
  962    1.234182e+01    8.541055e-01
* time: 18.89299988746643
  963    1.234175e+01    8.547420e-01
* time: 18.907999992370605
  964    1.234167e+01    8.541048e-01
* time: 18.913000106811523
  965    1.234160e+01    8.547401e-01
* time: 18.92799997329712
  966    1.234153e+01    8.541041e-01
* time: 18.93499994277954
  967    1.234146e+01    8.547382e-01
* time: 18.95799994468689
  968    1.234138e+01    8.541034e-01
* time: 18.996000051498413
  969    1.234131e+01    8.547363e-01
* time: 19.02900004386902
  970    1.234124e+01    8.541027e-01
* time: 19.033999919891357
  971    1.234117e+01    8.547344e-01
* time: 19.075999975204468
  972    1.234109e+01    8.541020e-01
* time: 19.085000038146973
  973    1.234102e+01    8.547325e-01
* time: 19.09599995613098
  974    1.234095e+01    8.541013e-01
* time: 19.125
  975    1.234087e+01    8.547306e-01
* time: 19.1489999294281
  976    1.234080e+01    8.541006e-01
* time: 19.163000106811523
  977    1.234073e+01    8.547287e-01
* time: 19.174999952316284
  978    1.234066e+01    8.540999e-01
* time: 19.194999933242798
  979    1.234058e+01    8.547268e-01
* time: 19.210999965667725
  980    1.234051e+01    8.540992e-01
* time: 19.227999925613403
  981    1.234044e+01    8.547249e-01
* time: 19.259999990463257
  982    1.234037e+01    8.540985e-01
* time: 19.35199999809265
  983    1.234029e+01    8.547230e-01

```

```

* time: 19.37999987602234
  984    1.234022e+01    8.540978e-01
* time: 19.388999938964844
  985    1.234015e+01    8.547211e-01
* time: 19.39800000190735
  986    1.234008e+01    8.540971e-01
* time: 19.40499997138977
  987    1.234000e+01    8.547192e-01
* time: 19.430999994277954
  988    1.233993e+01    8.540964e-01
* time: 19.448999881744385
  989    1.233986e+01    8.547173e-01
* time: 19.47499990463257
  990    1.233979e+01    8.540957e-01
* time: 19.506999969482422
  991    1.233971e+01    8.547155e-01
* time: 19.538000106811523
  992    1.233964e+01    8.540950e-01
* time: 19.57100009918213
  993    1.233957e+01    8.547136e-01
* time: 19.59999990463257
  994    1.233950e+01    8.540943e-01
* time: 19.608999967575073
  995    1.233942e+01    8.547117e-01
* time: 19.628999948501587
  996    1.233935e+01    8.540936e-01
* time: 19.634000062942505
  997    1.233928e+01    8.547098e-01
* time: 19.657999992370605
  998    1.233921e+01    8.540929e-01
* time: 19.661999940872192
  999    1.233913e+01    8.547079e-01
* time: 19.66700005531311
 1000    1.233906e+01    8.540922e-01
* time: 19.67300009727478

```

[8]: * Status: failure (reached maximum number of iterations) (line search failed)

```

* Candidate solution
  Minimizer: [4.51e+00, 2.04e+01]
  Minimum:   1.233906e+01

* Found with
  Algorithm:   Gradient Descent
  Initial Point: [2.00e+01, 2.00e+01]

* Convergence measures

```

```

|x - x'|           = 1.04e-04  0.0e+00
|x - x'|/|x'|     = 5.09e-06  0.0e+00
|f(x) - f(x')|    = 7.26e-05  0.0e+00
|f(x) - f(x')|/|f(x')| = 5.88e-06  0.0e+00
|g(x)|            = 8.54e-01  1.0e-06

```

* Work counters

```

Seconds run:  20 (vs limit Inf)
Iterations:   1000
f(x) calls:  2506
f(x) calls:  2506

```

```

[10]: res = optimize(rosenbrock, rosenbrock_gradient!,
                    [20.0, 20.0],
                    Optim.BFGS(),
                    Optim.Options(g_tol = 1e-6,
                                   store_trace = true,
                                   show_trace = true))

```

```

Iter      Function value      Gradient norm
  0      1.444036e+07      3.040038e+06
* time: 0.0
  1      2.680958e+06      8.881503e+05
* time: 0.014999866485595703
  2      6.650571e+05      5.858550e+05
* time: 0.01699995994567871
  3      2.806304e+02      7.167345e+02
* time: 0.03399991989135742
  4      2.794972e+02      3.772877e+01
* time: 0.04799985885620117
  5      2.694629e+02      8.824642e+02
* time: 0.06099987030029297
  6      2.679257e+02      1.073784e+03
* time: 0.07499980926513672
  7      2.615432e+02      1.595776e+03
* time: 0.11699986457824707
  8      2.480323e+02      1.178047e+03
* time: 0.12099981307983398
  9      2.411404e+02      6.281265e+02
* time: 0.12599992752075195
 10      2.318473e+02      2.328093e+02
* time: 0.14699983596801758
 11      2.293179e+02      7.429504e+02
* time: 0.18599987030029297
 12      2.271862e+02      1.282135e+03
* time: 0.20199990272521973

```

13	2.195866e+02	1.511525e+03
* time:	0.2089998722076416	
14	1.916722e+02	1.049794e+03
* time:	0.2369999885559082	
15	1.852938e+02	2.678043e+02
* time:	0.24699997901916504	
16	1.824770e+02	2.257172e+02
* time:	0.25999999046325684	
17	1.807223e+02	7.043838e+02
* time:	0.2890000343322754	
18	1.791228e+02	1.041848e+03
* time:	0.3399999141693115	
19	1.745125e+02	1.176254e+03
* time:	0.35500001907348633	
20	1.623655e+02	5.553827e+02
* time:	0.35899996757507324	
21	1.575354e+02	1.047299e+03
* time:	0.372999906539917	
22	1.438367e+02	7.248901e+02
* time:	0.3859999179840088	
23	1.415012e+02	7.714994e+01
* time:	0.4010000228881836	
24	1.361534e+02	2.564028e+02
* time:	0.4199998378753662	
25	1.343146e+02	4.888679e+02
* time:	0.43599987030029297	
26	1.322873e+02	7.347831e+02
* time:	0.46000003814697266	
27	1.292766e+02	9.784481e+02
* time:	0.4629998207092285	
28	1.175307e+02	6.237382e+02
* time:	0.46700000762939453	
29	1.147999e+02	1.772825e+02
* time:	0.48599982261657715	
30	1.085562e+02	5.876452e+02
* time:	0.49000000953674316	
31	1.068158e+02	8.555388e+02
* time:	0.5089998245239258	
32	9.705503e+01	2.918149e+02
* time:	0.5130000114440918	
33	9.558728e+01	4.672275e+02
* time:	0.5269999504089355	
34	9.221983e+01	5.120251e+02
* time:	0.5439999103546143	
35	8.630421e+01	5.578439e+02
* time:	0.556999921798706	
36	7.417278e+01	4.884472e+02
* time:	0.5639998912811279	

37	7.030566e+01	8.957325e+01
* time:	0.5779998302459717	
38	6.917628e+01	7.484095e+01
* time:	0.5929999351501465	
39	6.812603e+01	2.930674e+02
* time:	0.6329998970031738	
40	6.744612e+01	4.813804e+02
* time:	0.6469998359680176	
41	6.395227e+01	5.974543e+02
* time:	0.6949999332427979	
42	6.268214e+01	1.017006e+03
* time:	0.7259998321533203	
43	4.940524e+01	2.791254e+02
* time:	0.7289998531341553	
44	4.851947e+01	5.691400e+02
* time:	0.7439999580383301	
45	4.687109e+01	1.641540e+02
* time:	0.7730000019073486	
46	4.613982e+01	3.026013e+02
* time:	0.7909998893737793	
47	4.503657e+01	3.916941e+02
* time:	0.8369998931884766	
48	4.208788e+01	3.507581e+02
* time:	0.8550000190734863	
49	3.174450e+01	1.637115e+01
* time:	0.8589999675750732	
50	3.163005e+01	1.074344e+01
* time:	0.871999979019165	
51	3.035454e+01	2.248662e+02
* time:	0.8969998359680176	
52	2.918640e+01	2.716024e+02
* time:	0.8989999294281006	
53	2.578441e+01	1.306074e+02
* time:	0.9019999504089355	
54	2.451655e+01	2.335716e+02
* time:	0.9140000343322754	
55	2.097330e+01	2.137949e+02
* time:	0.9279999732971191	
56	1.996443e+01	9.432586e+00
* time:	0.9459998607635498	
57	1.817363e+01	1.501186e+02
* time:	0.9499998092651367	
58	1.762039e+01	2.070211e+02
* time:	0.9629998207092285	
59	1.499259e+01	8.796077e+01
* time:	0.9879999160766602	
60	1.415933e+01	1.589558e+02
* time:	1.0299999713897705	

61	1.180057e+01	1.263296e+02
* time:	1.0439999103546143	
62	1.121505e+01	9.872709e+00
* time:	1.0699999332427979	
63	1.004045e+01	7.453352e+01
* time:	1.0950000286102295	
64	9.867001e+00	9.258542e+01
* time:	1.120999813079834	
65	9.164464e+00	1.310284e+02
* time:	1.1419999599456787	
66	6.463125e+00	5.063399e+01
* time:	1.1689999103546143	
67	6.197393e+00	2.385676e+01
* time:	1.1979999542236328	
68	5.275474e+00	5.975018e+01
* time:	1.2139999866485596	
69	5.102342e+00	8.160447e+01
* time:	1.2170000076293945	
70	4.153169e+00	4.821250e+01
* time:	1.2319998741149902	
71	3.342622e+00	1.678964e+01
* time:	1.244999885559082	
72	3.296727e+00	2.268162e+01
* time:	1.2809998989105225	
73	3.091420e+00	2.716177e+01
* time:	1.2960000038146973	
74	2.903246e+00	3.176082e+01
* time:	1.312000036239624	
75	2.638841e+00	3.872453e+01
* time:	1.315999984741211	
76	2.267693e+00	5.239069e+01
* time:	1.319000005722046	
77	1.963828e+00	7.214768e+01
* time:	1.3359999656677246	
78	7.782322e-01	3.728406e+00
* time:	1.3609998226165771	
79	7.607595e-01	9.788744e+00
* time:	1.366999864578247	
80	7.315736e-01	1.750990e+01
* time:	1.375	
81	6.551173e-01	2.234978e+01
* time:	1.4040000438690186	
82	4.444449e-01	1.537023e+01
* time:	1.4229998588562012	
83	3.514617e-01	7.557805e+00
* time:	1.428999900817871	
84	2.247160e-01	7.153232e+00
* time:	1.444000005722046	


```

      85      2.115741e-01      9.854675e+00
* time: 1.4619998931884766
      86      1.589652e-01      9.905336e+00
* time: 1.4800000190734863
      87      5.498147e-02      1.425524e+00
* time: 1.4839999675750732
      88      4.919028e-02      3.049777e+00
* time: 1.498999834060669
      89      3.177471e-02      4.837723e+00
* time: 1.503000020980835
      90      5.392510e-04      6.955001e-01
* time: 1.5299999713897705
      91      1.076257e-04      1.165555e-01
* time: 1.5339999198913574
      92      3.683156e-05      1.419881e-01
* time: 1.5349998474121094
      93      7.641772e-07      3.009222e-02
* time: 1.5409998893737793
      94      2.553693e-11      8.507457e-05
* time: 1.5449998378753662
      95      8.109975e-17      3.265151e-07
* time: 1.5729999542236328

```

[10]: * Status: success

* Candidate solution

Minimizer: [1.00e+00, 1.00e+00]

Minimum: 8.109975e-17

* Found with

Algorithm: BFGS

Initial Point: [2.00e+01, 2.00e+01]

* Convergence measures

|x - x'| = 9.19e-06 0.0e+00

|x - x'|/|x'| = 9.19e-06 0.0e+00

|f(x) - f(x')| = 2.55e-11 0.0e+00

|f(x) - f(x')|/|f(x')| = 3.15e+05 0.0e+00

|g(x)| = 3.27e-07 1.0e-06

* Work counters

Seconds run: 2 (vs limit Inf)

Iterations: 95

f(x) calls: 284

f(x) calls: 284

```
[11]: using BenchmarkTools

@btime res = optimize(rosenbrock, rosenbrock_gradient!,
                      [0.0, 0.0], Optim.BFGS(),
                      Optim.Options(g_tol = 1e-6,
                                    store_trace = true,
                                    show_trace = false))
```

36.800 s (558 allocations: 29.64 KiB)

```
[11]: * Status: success

* Candidate solution
  Minimizer: [1.00e+00, 1.00e+00]
  Minimum:   7.645563e-21

* Found with
  Algorithm:   BFGS
  Initial Point: [0.00e+00, 0.00e+00]

* Convergence measures
  |x - x'|           = 3.48e-07  0.0e+00
  |x - x'|/|x'|     = 3.48e-07  0.0e+00
  |f(x) - f(x')|    = 6.91e-14  0.0e+00
  |f(x) - f(x')|/|f(x')| = 9.03e+06  0.0e+00
  |g(x)|           = 2.32e-09  1.0e-06

* Work counters
  Seconds run:   0 (vs limit Inf)
  Iterations:   16
  f(x) calls:   53
  f(x) calls:   53
```

```
[12]: iter = Optim.trace(res)
```

```
[12]: 96-element Array{OptimizationState{Float64,BFGS{LineSearches.InitialStatic{Float64},LineSearches.HagerZhang{Float64,Base.RefValue{Bool}}},Nothing,Nothing,Flat}},1}:
  0      1.444036e+07      3.040038e+06
* time: 0.0

  1      2.680958e+06      8.881503e+05
* time: 0.014999866485595703

  2      6.650571e+05      5.858550e+05
* time: 0.01699995994567871
```

3 2.806304e+02 7.167345e+02
* time: 0.03399991989135742

4 2.794972e+02 3.772877e+01
* time: 0.04799985885620117

5 2.694629e+02 8.824642e+02
* time: 0.06099987030029297

6 2.679257e+02 1.073784e+03
* time: 0.07499980926513672

7 2.615432e+02 1.595776e+03
* time: 0.11699986457824707

8 2.480323e+02 1.178047e+03
* time: 0.12099981307983398

9 2.411404e+02 6.281265e+02
* time: 0.12599992752075195

10 2.318473e+02 2.328093e+02
* time: 0.14699983596801758

11 2.293179e+02 7.429504e+02
* time: 0.18599987030029297

12 2.271862e+02 1.282135e+03
* time: 0.20199990272521973

84 2.247160e-01 7.153232e+00
* time: 1.444000005722046

85 2.115741e-01 9.854675e+00
* time: 1.4619998931884766

86 1.589652e-01 9.905336e+00
* time: 1.4800000190734863

87 5.498147e-02 1.425524e+00
* time: 1.4839999675750732

88 4.919028e-02 3.049777e+00
* time: 1.498999834060669

```

      89      3.177471e-02      4.837723e+00
* time: 1.503000020980835

      90      5.392510e-04      6.955001e-01
* time: 1.5299999713897705

      91      1.076257e-04      1.165555e-01
* time: 1.5339999198913574

      92      3.683156e-05      1.419881e-01
* time: 1.5349998474121094

      93      7.641772e-07      3.009222e-02
* time: 1.5409998893737793

      94      2.553693e-11      8.507457e-05
* time: 1.5449998378753662

      95      8.109975e-17      3.265151e-07
* time: 1.5729999542236328

```

1.1 Differentiation in Julia

Computing gradient and Hessian of complicated, and even sometimes simple, functions can be tedious. In order to alleviate this burden, it is possible to use numerical derivatives or automatic differentiation.

1.1.1 Numerical derivatives

Numerical derivatives function are provided in the package `Calculus`, as illustrated below.

```
[13]: using Calculus, LinearAlgebra
      rg = Calculus.gradient(rosenbrock)
```

```
[13]: #2 (generic function with 1 method)
```

Let's evaluate the newly constructed gradient function at the solution `[1,1]`.

```
[14]: gsol = rg([1,1])
```

```
[14]: 2-element Array{Float64,1}:
      1.4667356107373247e-8
     -1.1102239280930583e-14
```

We are close to zero, but there are approximation errors, that can prevent the convergence to the right solution, or at least impact the solution accuracy, as

```
[15]: norm(gsol)
```

```
[15]: 1.466735610737745e-8
```

```
[16]: storage = [0.0,0.0]
function rg!(storage::Vector, x::Vector)
    s = rg(x)
    storage[1:length(s)] = s[1:length(s)]
end
```

```
[16]: rg! (generic function with 1 method)
```

```
[17]: storage
```

```
[17]: 2-element Array{Float64,1}:
 0.0
 0.0
```

```
[18]: @btime res = optimize(rosenbrock, rg!,
    [0.0, 0.0],
    Optim.BFGS(),
    Optim.Options(g_tol = 1e-12,
        store_trace = true,
        show_trace = false))
```

```
62.899 s (803 allocations: 44.81 KiB)
```

```
[18]: * Status: success
```

```
* Candidate solution
```

```
Minimizer: [1.00e+00, 1.00e+00]
```

```
Minimum: 5.378308e-17
```

```
* Found with
```

```
Algorithm: BFGS
```

```
Initial Point: [0.00e+00, 0.00e+00]
```

```
* Convergence measures
```

```
|x - x'| = 1.32e-10 0.0e+00
```

```
|x - x'|/|x'| = 1.32e-10 0.0e+00
```

```
|f(x) - f(x')| = 9.31e-19 0.0e+00
```

```
|f(x) - f(x')|/|f(x')| = 1.73e-02 0.0e+00
```

```
|g(x)| = 2.78e-13 1.0e-12
```

```
* Work counters
```

```
Seconds run: 0 (vs limit Inf)
```

```
Iterations: 17
```

```
f(x) calls: 55
f(x) calls: 55
```

```
[22]: res = optimize(rosenbrock, rg!,
                    [1.1, 1.1],
                    Optim.GradientDescent(),
                    Optim.Options(g_tol = 1e-12,
                                  store_trace = true,
                                  show_trace = true))
```

```
Iter      Function value  Gradient norm
  0      1.220000e+00    4.860000e+01
* time: 0.0009999275207519531
  1      5.804214e-03    2.125407e+00
* time: 0.03600001335144043
  2      3.342431e-03    4.583199e-02
* time: 0.0409998893737793
  3      3.025234e-03    7.072574e-01
* time: 0.06200003623962402
  4      2.774066e-03    1.561595e-01
* time: 0.10399985313415527
  5      2.754813e-03    3.619916e-02
* time: 0.1099998950958252
  6      2.735659e-03    1.553929e-01
* time: 0.1419999599456787
  7      2.716594e-03    3.619630e-02
* time: 0.1549999713897705
  8      2.695676e-03    1.642356e-01
* time: 0.18599987030029297
  9      2.674873e-03    3.622467e-02
* time: 0.20799994468688965
 10      2.651249e-03    1.769096e-01
* time: 0.21499991416931152
 11      2.627785e-03    3.631260e-02
* time: 0.24199986457824707
 12      2.599317e-03    1.979975e-01
* time: 0.2739999294281006
 13      2.571002e-03    3.655250e-02
* time: 0.28299999237060547
 14      2.530203e-03    2.437361e-01
* time: 0.2869999408721924
 15      2.489586e-03    3.738017e-02
* time: 0.30299997329711914
 16      2.334856e-03    5.027984e-01
* time: 0.33699989318847656
 17      2.180480e-03    5.772634e-02
```

```

* time: 0.35699987411499023
  18    2.177342e-03    4.660964e-02
* time: 0.37800002098083496
  19    2.174207e-03    5.765494e-02
* time: 0.41100001335144043
  20    2.171076e-03    4.653512e-02
* time: 0.4289999008178711
  21    2.167949e-03    5.758317e-02
* time: 0.4500000476837158
  22    2.164826e-03    4.646069e-02
* time: 0.4549999237060547
  23    2.161707e-03    5.751145e-02
* time: 0.45799994468688965
  24    2.158592e-03    4.638637e-02
* time: 0.4789998531341553
  25    2.155481e-03    5.743979e-02
* time: 0.5190000534057617
  26    2.152374e-03    4.631215e-02
* time: 0.5230000019073486
  27    2.149271e-03    5.736818e-02
* time: 0.5350000858306885
  28    2.146171e-03    4.623803e-02
* time: 0.5460000038146973
  29    2.143076e-03    5.729662e-02
* time: 0.6039998531341553
  30    2.139984e-03    4.616401e-02
* time: 0.6399998664855957
  31    2.136897e-03    5.722512e-02
* time: 0.6440000534057617
  32    2.133813e-03    4.609008e-02
* time: 0.6649999618530273
  33    2.130733e-03    5.715367e-02
* time: 0.6700000762939453
  34    2.127657e-03    4.601626e-02
* time: 0.6740000247955322
  35    2.124585e-03    5.708228e-02
* time: 0.681999921798706
  36    2.121517e-03    4.594253e-02
* time: 0.7170000076293945
  37    2.118453e-03    5.701094e-02
* time: 0.7460000514984131
  38    2.115393e-03    4.586891e-02
* time: 0.7850000858306885
  39    2.112336e-03    5.693965e-02
* time: 0.8029999732971191
  40    2.109283e-03    4.579538e-02
* time: 0.8190000057220459
  41    2.106235e-03    5.686842e-02

```

```

* time: 0.8550000190734863
  42    2.103190e-03    4.572195e-02
* time: 0.878000020980835
  43    2.100149e-03    5.679724e-02
* time: 0.8899998664855957
  44    2.097112e-03    4.564862e-02
* time: 0.8980000019073486
  45    2.094078e-03    5.672612e-02
* time: 0.9140000343322754
  46    2.091049e-03    4.557539e-02
* time: 0.9389998912811279
  47    2.088023e-03    5.665505e-02
* time: 0.9689998626708984
  48    2.085001e-03    4.550226e-02
* time: 1.0339999198913574
  49    2.081984e-03    5.658403e-02
* time: 1.069000005722046
  50    2.078969e-03    4.542923e-02
* time: 1.121000051498413
  51    2.075959e-03    5.651306e-02
* time: 1.1399998664855957
  52    2.072953e-03    4.535630e-02
* time: 1.2070000171661377
  53    2.069950e-03    5.644215e-02
* time: 1.2379999160766602
  54    2.066951e-03    4.528346e-02
* time: 1.251999855041504
  55    2.063956e-03    5.637130e-02
* time: 1.2790000438690186
  56    2.060965e-03    4.521072e-02
* time: 1.315999984741211
  57    2.057978e-03    5.630050e-02
* time: 1.319000005722046
  58    2.054994e-03    4.513809e-02
* time: 1.3399999141693115
  59    2.052014e-03    5.622975e-02
* time: 1.3539998531341553
  60    2.049038e-03    4.506555e-02
* time: 1.36899995803833
  61    2.046066e-03    5.615905e-02
* time: 1.3999998569488525
  62    2.043097e-03    4.499310e-02
* time: 1.430999994277954
  63    2.040133e-03    5.608841e-02
* time: 1.4670000076293945
  64    2.037172e-03    4.492076e-02
* time: 1.496999979019165
  65    2.034215e-03    5.601782e-02

```



```

* time: 1.5709998607635498
  66      2.031261e-03      4.484851e-02
* time: 1.6029999256134033
  67      2.028312e-03      5.594729e-02
* time: 1.6989998817443848
  68      2.025366e-03      4.477636e-02
* time: 1.7390000820159912
  69      2.022424e-03      5.587681e-02
* time: 1.7979998588562012
  70      2.019485e-03      4.470431e-02
* time: 1.8239998817443848
  71      2.016551e-03      5.580638e-02
* time: 1.8380000591278076
  72      2.013620e-03      4.463236e-02
* time: 1.8989999294281006
  73      2.010693e-03      5.573601e-02
* time: 2.1010000705718994
  74      2.007769e-03      4.456051e-02
* time: 2.1050000190734863
  75      2.004850e-03      5.566569e-02
* time: 2.11899995803833
  76      2.001934e-03      4.448875e-02
* time: 2.122999906539917
  77      1.999022e-03      5.559543e-02
* time: 2.138000011444092
  78      1.996113e-03      4.441709e-02
* time: 2.1589999198913574
  79      1.993208e-03      5.552522e-02
* time: 2.1670000553131104
  80      1.990307e-03      4.434553e-02
* time: 2.2039999961853027
  81      1.987410e-03      5.545506e-02
* time: 2.2189998626708984
  82      1.984516e-03      4.427406e-02
* time: 2.2269999980926514
  83      1.981626e-03      5.538496e-02
* time: 2.247999906539917
  84      1.978740e-03      4.420269e-02
* time: 2.2669999599456787
  85      1.975858e-03      5.531491e-02
* time: 2.2720000743865967
  86      1.972979e-03      4.413142e-02
* time: 2.2779998779296875
  87      1.970104e-03      5.524491e-02
* time: 2.316999912261963
  88      1.967232e-03      4.406025e-02
* time: 2.3489999771118164
  89      1.964364e-03      5.517497e-02

```

```

* time: 2.3539998531341553
  90    1.961500e-03    4.398917e-02
* time: 2.3569998741149902
  91    1.958640e-03    5.510508e-02
* time: 2.374000072479248
  92    1.955783e-03    4.391819e-02
* time: 2.3929998874664307
  93    1.952930e-03    5.503525e-02
* time: 2.4110000133514404
  94    1.950080e-03    4.384731e-02
* time: 2.437000036239624
  95    1.947234e-03    5.496547e-02
* time: 2.4559998512268066
  96    1.944392e-03    4.377653e-02
* time: 2.4719998836517334
  97    1.941554e-03    5.489574e-02
* time: 2.4789998531341553
  98    1.938718e-03    4.370584e-02
* time: 2.49399995803833
  99    1.935887e-03    5.482606e-02
* time: 2.501999855041504
 100    1.933060e-03    4.363524e-02
* time: 2.5220000743865967
 101    1.930236e-03    5.475645e-02
* time: 2.558000087738037
 102    1.927415e-03    4.356475e-02
* time: 2.5959999561309814
 103    1.924598e-03    5.468688e-02
* time: 2.6080000400543213
 104    1.921785e-03    4.349435e-02
* time: 2.624000072479248
 105    1.918976e-03    5.461737e-02
* time: 2.6389999389648438
 106    1.916170e-03    4.342404e-02
* time: 2.6559998989105225
 107    1.913368e-03    5.454791e-02
* time: 2.672999858856201
 108    1.910569e-03    4.335384e-02
* time: 2.690000057220459
 109    1.907774e-03    5.447850e-02
* time: 2.7100000381469727
 110    1.904982e-03    4.328373e-02
* time: 2.7170000076293945
 111    1.902194e-03    5.440915e-02
* time: 2.7249999046325684
 112    1.899410e-03    4.321371e-02
* time: 2.7279999256134033
 113    1.896629e-03    5.433986e-02

```

```

* time: 2.746999979019165
  114    1.893852e-03    4.314379e-02
* time: 2.7649998664855957
  115    1.891078e-03    5.427061e-02
* time: 2.7990000247955322
  116    1.888308e-03    4.307397e-02
* time: 2.828000068664551
  117    1.885542e-03    5.420142e-02
* time: 2.8480000495910645
  118    1.882779e-03    4.300424e-02
* time: 2.88700008392334
  119    1.880020e-03    5.413229e-02
* time: 2.8910000324249268
  120    1.877264e-03    4.293461e-02
* time: 2.8980000019073486
  121    1.874512e-03    5.406321e-02
* time: 2.927999973297119
  122    1.871763e-03    4.286507e-02
* time: 2.947999954223633
  123    1.869018e-03    5.399418e-02
* time: 2.99399995803833
  124    1.866276e-03    4.279563e-02
* time: 3.010999917984009
  125    1.863538e-03    5.392520e-02
* time: 3.0199999809265137
  126    1.860803e-03    4.272629e-02
* time: 3.0239999294281006
  127    1.858073e-03    5.385628e-02
* time: 3.0320000648498535
  128    1.855345e-03    4.265704e-02
* time: 3.059000015258789
  129    1.852621e-03    5.378742e-02
* time: 3.072999954223633
  130    1.849901e-03    4.258788e-02
* time: 3.101999980926514
  131    1.847184e-03    5.371860e-02
* time: 3.119999885559082
  132    1.844470e-03    4.251883e-02
* time: 3.138000011444092
  133    1.841761e-03    5.364985e-02
* time: 3.1710000038146973
  134    1.839054e-03    4.244986e-02
* time: 3.187999963760376
  135    1.836351e-03    5.358114e-02
* time: 3.2170000076293945
  136    1.833652e-03    4.238099e-02
* time: 3.2309999465942383
  137    1.830956e-03    5.351249e-02

```

```

* time: 3.253999948501587
  138    1.828263e-03    4.231222e-02
* time: 3.259000062942505
  139    1.825575e-03    5.344389e-02
* time: 3.2850000858306885
  140    1.822889e-03    4.224354e-02
* time: 3.313999891281128
  141    1.820207e-03    5.337535e-02
* time: 3.338999866485596
  142    1.817529e-03    4.217496e-02
* time: 3.3919999599456787
  143    1.814854e-03    5.330686e-02
* time: 3.4179999828338623
  144    1.812182e-03    4.210647e-02
* time: 3.4739999771118164
  145    1.809514e-03    5.323842e-02
* time: 3.484999895095825
  146    1.806849e-03    4.203807e-02
* time: 3.501999855041504
  147    1.804188e-03    5.317004e-02
* time: 3.5279998779296875
  148    1.801530e-03    4.196977e-02
* time: 3.5399999618530273
  149    1.798876e-03    5.310171e-02
* time: 3.5429999828338623
  150    1.796225e-03    4.190156e-02
* time: 3.5929999351501465
  151    1.793578e-03    5.303343e-02
* time: 3.61899995803833
  152    1.790934e-03    4.183345e-02
* time: 3.6449999809265137
  153    1.788293e-03    5.296521e-02
* time: 3.6700000762939453
  154    1.785656e-03    4.176543e-02
* time: 3.697000026702881
  155    1.783023e-03    5.289705e-02
* time: 3.7260000705718994
  156    1.780392e-03    4.169751e-02
* time: 3.739000082015991
  157    1.777766e-03    5.282893e-02
* time: 3.7649998664855957
  158    1.775142e-03    4.162968e-02
* time: 3.7809998989105225
  159    1.772522e-03    5.276087e-02
* time: 3.816999912261963
  160    1.769905e-03    4.156195e-02
* time: 3.825000047683716
  161    1.767292e-03    5.269287e-02

```

```

* time: 3.8269999027252197
  162    1.764682e-03    4.149431e-02
* time: 3.8439998626708984
  163    1.762076e-03    5.262491e-02
* time: 3.8580000400543213
  164    1.759473e-03    4.142676e-02
* time: 3.878000020980835
  165    1.756874e-03    5.255702e-02
* time: 3.8949999809265137
  166    1.754277e-03    4.135930e-02
* time: 3.9029998779296875
  167    1.751684e-03    5.248917e-02
* time: 3.9070000648498535
  168    1.749095e-03    4.129194e-02
* time: 3.921999931335449
  169    1.746509e-03    5.242138e-02
* time: 3.950000047683716
  170    1.743926e-03    4.122468e-02
* time: 3.9570000171661377
  171    1.741347e-03    5.235364e-02
* time: 3.984999895095825
  172    1.738771e-03    4.115750e-02
* time: 3.98799991607666
  173    1.736198e-03    5.228596e-02
* time: 3.990999937057495
  174    1.733629e-03    4.109042e-02
* time: 3.99399995803833
  175    1.731063e-03    5.221833e-02
* time: 4.014999866485596
  176    1.728500e-03    4.102344e-02
* time: 4.0329999923706055
  177    1.725941e-03    5.215076e-02
* time: 4.050999879837036
  178    1.723385e-03    4.095654e-02
* time: 4.069000005722046
  179    1.720833e-03    5.208324e-02
* time: 4.088000059127808
  180    1.718283e-03    4.088974e-02
* time: 4.116999864578247
  181    1.715737e-03    5.201577e-02
* time: 4.144000053405762
  182    1.713195e-03    4.082304e-02
* time: 4.158999919891357
  183    1.710656e-03    5.194835e-02
* time: 4.167999982833862
  184    1.708120e-03    4.075642e-02
* time: 4.185999870300293
  185    1.705587e-03    5.188099e-02

```

```

* time: 4.194000005722046
  186    1.703058e-03    4.068990e-02
* time: 4.216000080108643
  187    1.700532e-03    5.181369e-02
* time: 4.23199987411499
  188    1.698009e-03    4.062347e-02
* time: 4.244999885559082
  189    1.695490e-03    5.174644e-02
* time: 4.260999917984009
  190    1.692973e-03    4.055713e-02
* time: 4.283999919891357
  191    1.690461e-03    5.167924e-02
* time: 4.308000087738037
  192    1.687951e-03    4.049089e-02
* time: 4.322000026702881
  193    1.685445e-03    5.161209e-02
* time: 4.325999975204468
  194    1.682942e-03    4.042474e-02
* time: 4.328999996185303
  195    1.680442e-03    5.154500e-02
* time: 4.349999904632568
  196    1.677946e-03    4.035868e-02
* time: 4.368000030517578
  197    1.675453e-03    5.147797e-02
* time: 4.381999969482422
  198    1.672963e-03    4.029271e-02
* time: 4.38700008392334
  199    1.670477e-03    5.141098e-02
* time: 4.389999866485596
  200    1.667993e-03    4.022684e-02
* time: 4.4059998989105225
  201    1.665513e-03    5.134406e-02
* time: 4.421999931335449
  202    1.663036e-03    4.016106e-02
* time: 4.453000068664551
  203    1.660563e-03    5.127718e-02
* time: 4.50600004196167
  204    1.658092e-03    4.009537e-02
* time: 4.542999982833862
  205    1.655625e-03    5.121036e-02
* time: 4.559000015258789
  206    1.653162e-03    4.002977e-02
* time: 4.575000047683716
  207    1.650701e-03    5.114359e-02
* time: 4.582000017166138
  208    1.648244e-03    3.996426e-02
* time: 4.5950000286102295
  209    1.645790e-03    5.107688e-02

```

```

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* time: 4.683000087738037
  212    1.638447e-03    3.983352e-02
* time: 4.70799994468689
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* time: 4.710000038146973
  214    1.633567e-03    3.976829e-02
* time: 4.7170000076293945
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* time: 4.730999946594238
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* time: 4.75600004196167
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* time: 4.772000074386597
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* time: 4.7850000858306885
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* time: 4.802000045776367
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* time: 4.816999912261963
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* time: 4.822999954223633
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* time: 4.82699990272522
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* time: 4.841000080108643
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* time: 4.85699987411499
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* time: 4.859999895095825
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* time: 4.88700008392334
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* time: 4.901000022888184
  228    1.599771e-03    3.931423e-02
* time: 4.914999961853027
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* time: 4.924000024795532
  230    1.594993e-03    3.924973e-02
* time: 4.960999965667725
  231    1.592609e-03    5.034660e-02
* time: 4.978999853134155
  232    1.590228e-03    3.918531e-02
* time: 5.009000062942505
  233    1.587851e-03    5.028053e-02

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* time: 5.046999931335449
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* time: 5.075000047683716
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* time: 5.0899999141693115
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* time: 5.119999885559082
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* time: 5.124000072479248
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* time: 5.138999938964844
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* time: 5.1570000648498535
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* time: 5.16700005531311
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* time: 5.181999921798706
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* time: 5.187000036239624
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* time: 5.191999912261963
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* time: 5.241999864578247
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* time: 5.253999948501587
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* time: 5.2829999923706055
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* time: 5.299999952316284
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* time: 5.32099986076355
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* time: 5.348999977111816
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* time: 5.352999925613403
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* time: 5.372999906539917
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* time: 5.38700008392334
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* time: 5.4049999713897705
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* time: 5.489000082015991
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* time: 5.496000051498413
  262    1.520253e-03    3.822997e-02
* time: 5.5329999923706055
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* time: 5.560999870300293
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* time: 5.579999923706055
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* time: 5.632999897003174
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* time: 5.6549999713897705
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* time: 5.690999984741211
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* time: 5.704999923706055
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* time: 5.723999977111816
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* time: 5.73799991607666
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* time: 5.75600004196167
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* time: 5.773000001907349
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* time: 5.812000036239624
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* time: 5.83299994468689
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* time: 5.861000061035156
  278    1.484058e-03    3.772869e-02
* time: 5.879999876022339
  279    1.481821e-03    4.877593e-02
* time: 5.895999908447266
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* time: 6.009999990463257
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* time: 6.0290000438690186
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* time: 6.085000038146973
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* time: 6.109999895095825
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* time: 6.240000009536743
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* time: 6.254999876022339
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* time: 6.2779998779296875
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* time: 6.318000078201294
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* time: 6.323999881744385
  302    1.431197e-03    3.698738e-02
* time: 6.348999977111816
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* time: 6.376999855041504
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* time: 6.800999879837036
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* time: 6.948999881744385
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* time: 7.052000045776367
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* time: 7.072000026702881
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* time: 7.0980000495910645
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* time: 7.102999925613403
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* time: 7.116999864578247
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* time: 7.187999963760376
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* time: 7.366999864578247
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* time: 7.575000047683716
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* time: 7.63700008392334
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* time: 8.336999893188477
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* time: 8.83899998664856
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* time: 8.855999946594238
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* time: 8.927000045776367
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  927    5.287635e-04    3.064705e-02
* time: 19.197999954223633
  928    5.278818e-04    2.164275e-02
* time: 19.216000080108643
  929    5.270015e-04    3.059995e-02

```

```

* time: 19.269999980926514
  930    5.261225e-04    2.160457e-02
* time: 19.289000034332275
  931    5.252449e-04    3.055292e-02
* time: 19.296000003814697
  932    5.243686e-04    2.156645e-02
* time: 19.318000078201294
  933    5.234937e-04    3.050593e-02
* time: 19.335000038146973
  934    5.226201e-04    2.152839e-02
* time: 19.37999987602234
  935    5.217479e-04    3.045900e-02
* time: 19.40499997138977
  936    5.208770e-04    2.149038e-02
* time: 19.473999977111816
  937    5.200074e-04    3.041213e-02
* time: 19.51699995994568
  938    5.191391e-04    2.145244e-02
* time: 19.555999994277954
  939    5.182722e-04    3.036531e-02
* time: 19.598000049591064
  940    5.174066e-04    2.141456e-02
* time: 19.63700008392334
  941    5.165424e-04    3.031854e-02
* time: 19.66599988937378
  942    5.156795e-04    2.137674e-02
* time: 19.707000017166138
  943    5.148179e-04    3.027182e-02
* time: 19.730000019073486
  944    5.139576e-04    2.133898e-02
* time: 19.750999927520752
  945    5.130986e-04    3.022516e-02
* time: 19.802000045776367
  946    5.122410e-04    2.130128e-02
* time: 19.830999851226807
  947    5.113847e-04    3.017856e-02
* time: 19.866999864578247
  948    5.105297e-04    2.126364e-02
* time: 19.91000008583069
  949    5.096760e-04    3.013200e-02
* time: 19.944999933242798
  950    5.088236e-04    2.122605e-02
* time: 19.957000017166138
  951    5.079725e-04    3.008550e-02
* time: 19.986999988555908
  952    5.071228e-04    2.118853e-02
* time: 20.007999897003174
  953    5.062743e-04    3.003906e-02

```

```

* time: 20.06599998474121
  954    5.054272e-04    2.115107e-02
* time: 20.07800006866455
  955    5.045813e-04    2.999266e-02
* time: 20.098000049591064
  956    5.037368e-04    2.111367e-02
* time: 20.13599991798401
  957    5.028936e-04    2.994632e-02
* time: 20.200000047683716
  958    5.020516e-04    2.107632e-02
* time: 20.264999866485596
  959    5.012110e-04    2.990004e-02
* time: 20.38100004196167
  960    5.003716e-04    2.103904e-02
* time: 20.44099998474121
  961    4.995336e-04    2.985381e-02
* time: 20.506999969482422
  962    4.986968e-04    2.100181e-02
* time: 20.549000024795532
  963    4.978613e-04    2.980763e-02
* time: 20.575000047683716
  964    4.970271e-04    2.096465e-02
* time: 20.59999990463257
  965    4.961942e-04    2.976150e-02
* time: 20.65400004386902
  966    4.953626e-04    2.092754e-02
* time: 20.680999994277954
  967    4.945323e-04    2.971543e-02
* time: 20.7189998626709
  968    4.937032e-04    2.089049e-02
* time: 20.7739999294281
  969    4.928755e-04    2.966941e-02
* time: 20.81599998474121
  970    4.920489e-04    2.085350e-02
* time: 20.90499997138977
  971    4.912237e-04    2.962345e-02
* time: 20.990000009536743
  972    4.903998e-04    2.081657e-02
* time: 21.00499987602234
  973    4.895771e-04    2.957754e-02
* time: 21.032000064849854
  974    4.887557e-04    2.077970e-02
* time: 21.1019999809265
  975    4.879356e-04    2.953168e-02
* time: 21.122999906539917
  976    4.871167e-04    2.074289e-02
* time: 21.15400004386902
  977    4.862991e-04    2.948588e-02

```

```

* time: 21.194999933242798
  978    4.854827e-04    2.070613e-02
* time: 21.249000072479248
  979    4.846677e-04    2.944012e-02
* time: 21.289000034332275
  980    4.838538e-04    2.066944e-02
* time: 21.35199999809265
  981    4.830413e-04    2.939443e-02
* time: 21.378999948501587
  982    4.822300e-04    2.063280e-02
* time: 21.418999910354614
  983    4.814199e-04    2.934878e-02
* time: 21.453999996185303
  984    4.806111e-04    2.059622e-02
* time: 21.507999897003174
  985    4.798036e-04    2.930319e-02
* time: 21.575000047683716
  986    4.789973e-04    2.055970e-02
* time: 21.624000072479248
  987    4.781923e-04    2.925766e-02
* time: 21.664999961853027
  988    4.773884e-04    2.052324e-02
* time: 21.698999881744385
  989    4.765859e-04    2.921217e-02
* time: 21.72099995613098
  990    4.757846e-04    2.048684e-02
* time: 21.756999969482422
  991    4.749845e-04    2.916674e-02
* time: 21.83399987220764
  992    4.741857e-04    2.045049e-02
* time: 21.853999853134155
  993    4.733881e-04    2.912136e-02
* time: 21.91700005531311
  994    4.725917e-04    2.041421e-02
* time: 21.98799991607666
  995    4.717966e-04    2.907604e-02
* time: 22.03600001335144
  996    4.710027e-04    2.037798e-02
* time: 22.068000078201294
  997    4.702101e-04    2.903077e-02
* time: 22.092000007629395
  998    4.694187e-04    2.034180e-02
* time: 22.139999866485596
  999    4.686285e-04    2.898555e-02
* time: 22.20899987220764
 1000    4.678395e-04    2.030569e-02
* time: 22.249000072479248

```

```
[22]: * Status: failure (reached maximum number of iterations) (line search failed)

* Candidate solution
  Minimizer: [1.02e+00, 1.04e+00]
  Minimum:   4.678395e-04

* Found with
  Algorithm:   Gradient Descent
  Initial Point: [1.10e+00, 1.10e+00]

* Convergence measures
  |x - x'|           = 4.17e-05  0.0e+00
  |x - x'|/|x'|     = 3.99e-05  0.0e+00
  |f(x) - f(x')|    = 7.89e-07  0.0e+00
  |f(x) - f(x')|/|f(x')| = 1.69e-03  0.0e+00
  |g(x)|           = 2.03e-02  1.0e-12

* Work counters
  Seconds run:   22 (vs limit Inf)
  Iterations:   1000
  f(x) calls:   2515
  f(x) calls:   2515
```

1.1.2 Automatic differentiation

```
[23]: using Pkg
      Pkg.add("ForwardDiff")

Updating registry at
`C:\Users\slash\.julia\registries\General`
Updating git-repo
`https://github.com/JuliaRegistries/General.git`
[1mFetching: [=====>]
100.0 %.0 %.6 %.9 % [=====> ] 11.7 %2
%=====> ] 20.0 %=====>
] 23.3 %Fetching: [=====>
] 32.0 %Fetching: [=====>
] 38.3 %42.3 % [=====> ] 44.9 %
] 51.3 %=====> ] 52.8 % %3 %0 %.5 %72.1
%Fetching: [=====> ] 73.9
%.0 %.6 %.2 %> ] 78.1 % [=====> ]
79.3 %Fetching: [=====> ]
88.4 % [=====> ] 91.8 %> ] 94.2 %
[=====> ] 95.1 %] 97.9 %

Resolving package versions...
Installed NLSolversBase v7.6.1
```



```
Installed OpenBLAS_jll      v0.3.7+5
Installed MutableArithmetics v0.2.2
```

```
Updating
```

```
`C:\Users\slash\.julia\environments\v1.3\Project.toml`
[no changes]
```

```
Updating
```

```
`C:\Users\slash\.julia\environments\v1.3\Manifest.toml`
[d8a4904e] ↑ MutableArithmetics v0.2.1  v0.2.2
[d41bc354] ↑ NLSolversBase v7.6.0   v7.6.1
[4536629a] ↑ OpenBLAS_jll v0.3.7+4  v0.3.7+5
```

```
[24]: using ForwardDiff
```

```
g = x -> ForwardDiff.gradient(rosenbrock, x);
H = x -> ForwardDiff.hessian(rosenbrock, x)

function g!(storage::Vector, x::Vector)
    s = g(x)
    storage[1:length(s)] = s[1:length(s)]
end
```

```
[24]: g! (generic function with 1 method)
```

```
[25]: g([1.0,1.0])
```

```
[25]: 2-element Array{Float64,1}:
 -0.0
  0.0
```

```
[26]: res = optimize(rosenbrock, g!,
                    [0.0, 0.0],
                    Optim.BFGS(),
                    Optim.Options(g_tol = 1e-6,
                                   store_trace = true,
                                   show_trace = false))
```

```
[26]: * Status: success
```

```
* Candidate solution
```

```
Minimizer: [1.00e+00, 1.00e+00]
Minimum:   7.645684e-21
```

```
* Found with
```

```
Algorithm:   BFGS
Initial Point: [0.00e+00, 0.00e+00]
```

```
* Convergence measures
```

```

|x - x'|          = 3.48e-07  0.0e+00
|x - x'|/|x'|    = 3.48e-07  0.0e+00
|f(x) - f(x')|   = 6.91e-14  0.0e+00
|f(x) - f(x')|/|f(x')| = 9.03e+06  0.0e+00
|g(x)|           = 2.32e-09  1.0e-06

```

* Work counters

```

Seconds run:  0 (vs limit Inf)
Iterations:   16
f(x) calls:  53
f(x) calls:  53

```

```

[27]: @btime res = optimize(rosenbrock, g!,
    [0.0, 0.0],
    Optim.BFGS(),
    Optim.Options(g_tol = 1e-6,
        store_trace = true,
        show_trace = false))

```

79.900 s (876 allocations: 52.00 KiB)

[27]: * Status: success

* Candidate solution

```

Minimizer: [1.00e+00, 1.00e+00]
Minimum:   7.645684e-21

```

* Found with

```

Algorithm:   BFGS
Initial Point: [0.00e+00, 0.00e+00]

```

* Convergence measures

```

|x - x'|          = 3.48e-07  0.0e+00
|x - x'|/|x'|    = 3.48e-07  0.0e+00
|f(x) - f(x')|   = 6.91e-14  0.0e+00
|f(x) - f(x')|/|f(x')| = 9.03e+06  0.0e+00
|g(x)|           = 2.32e-09  1.0e-06

```

* Work counters

```

Seconds run:  0 (vs limit Inf)
Iterations:   16
f(x) calls:  53
f(x) calls:  53

```

1.2 Newton method

$$x_{k+1} = x_k - \nabla^2 f(x_k)^{-1} \nabla f(x_k)$$

or

$$\nabla^2 f(x_k) x_{k+1} = \nabla^2 f(x_k) x_k - \nabla f(x_k)$$

```
[28]: function Newton(f::Function, g::Function, h::Function,
    xstart::Vector, verbose::Bool = false,
    ::Float64 = 1e-6, nmax::Int64 = 1000)

    k = 1
    x = xstart
    n = length(x)
    2 = *
    H = zeros(n,n)+I
    dfx = ones(n)

    if (verbose)
        fx = f(x)
        println("$k. x = $x, f(x) = $fx")
    end

    g(dfx, x)

    while (dot(dfx,dfx) > 2 && k <= nmax)
        k += 1
        g(dfx,x)
        h(H,x)
        # Hs = -dfx, x_{k+1} = x_k - s # s = -inv(H)*dfx vs s = -H\dfx
        x -= H\dfx # x = x - s
        if (verbose)
            fx = f(x)
            println("$k. x = $x, f(x) = $fx ")
        end
    end
end
```

[28]: Newton (generic function with 4 methods)

```
[29]: Newton(rosenbrock, rosenbrock_gradient!, rosenbrock_hessian!, [-100.0,100.0],  
↳true)
```

1. x = [-100.0, 100.0], f(x) = 9.801010201e9
2. x = [-99.99994898992475, 9999.989797984948], f(x) = 10200.9896959674
3. x = [0.9999473760899633, -10199.97917119076], f(x) = 1.0405997390386257e10
4. x = [0.9999473761157568, 0.9998947549993318], f(x) = 2.7692731928427748e-9
5. x = [0.9999999999999847, 0.9999999972306961], f(x) = 7.668874361715257e-16

- 6. $x = [1.0, 1.0]$, $f(x) = 0.0$
- 7. $x = [1.0, 1.0]$, $f(x) = 0.0$

1.3 Implementation of a linesearch algorithm

A very basic linesearch implementation skeleton follows.

```
[38]: function ls(f::Function, g::Function, h:: Function,
               x0::Vector,
               direction::Function, steplength::Function,
               ::Float64 = 1e-6, nmax::Int64 = 1000)

    k = 0
    x = x0
    2 = *
    n = length(x)

    dfx = ones(n)

    g(dfx, x)

    # println("$k. $x")

    while (dot(dfx,dfx) > 2 && k <= nmax)
        # Compute the search direction
        d, dfx = direction(f,g,h,x)
        # Compute the step length along d
        = steplength(f,dfx,x,d)
        # Update the iterate
        x += *d
        k += 1
    #     println("$k. $x")
    end

    return x
end
```

[38]: ls (generic function with 3 methods)

```
[39]: constantStep(f::Function, dfx:: Vector, x:: Vector, d::Vector) = 1
```

[39]: constantStep (generic function with 1 method)

```
[42]: n = 3;
Matrix{Float64}(I,n,n)
```

```
[42]: 3×3 Array{Float64,2}:
 1.0  0.0  0.0
```

```
0.0 1.0 0.0
0.0 0.0 1.0
```

```
[41]: function direction(f::Function, g::Function, h::Function, x::Vector)
      n = length(x)
      df = ones(n)
      H = zeros(n,n)+I
      g(df,x)
      h(H,x)
      return -H\df, df
end
```

[41]: direction (generic function with 1 method)

```
[43]: x = ls(rosenbrock, rosenbrock_gradient!, rosenbrock_hessian!,
           [0.0,0.0], direction, constantStep)
```

[43]: 2-element Array{Float64,1}:
1.0
1.0

```
[46]: function ArmijoStep(f::Function, dfx::Vector, x::Vector, d::Vector,
      max::Float64 = 1.0, ::Float64 =0.1, ::Float64 =0.2)

      s = *dot(dfx,d)
      = max

      fx = f(x)
      fxcand = f(x+ *d)

      while (fxcand >= fx+ *s)
          *=
          fxcand = f(x+ *d)
      end

      return
end
```

[46]: ArmijoStep (generic function with 4 methods)

```
[47]: @btime x =ls(rosenbrock, rosenbrock_gradient!, rosenbrock_hessian!,
                [0.0,0.0], direction, ArmijoStep)
```

18.599 s (221 allocations: 18.58 KiB)

[47]: 2-element Array{Float64,1}:
0.99999999999999569

0.9999999999998284

```
[48]: norm(ones(2)-x)
```

[48]: 0.0

```
[49]: function direction3(f::Function, g:: Function, h:: Function, x::Vector)
      n = length(x)
      df = ones(n)
      H = zeros(n,n)+I
      g(df,x)
      h(H,x)
      H[1,2] = H[2,1]= 0.0
      return -H\df, df
end
```

[49]: direction3 (generic function with 1 method)

```
[50]: ls(rosenbrock, rosenbrock_gradient!, rosenbrock_hessian!,
      [0.0,0.0], direction3, ArmijoStep)
```

[50]: 2-element Array{Float64,1}:
0.9259531981936803
0.8567279574660482

```
[ ]:
```