kkt_secondorder

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1 KKT conditions

1.1 Second-order conditions

1.2 Example 1

A simple example, adapted from http://www.math.ubc.ca/~israel/m340/

Consider the problem

$$\begin{aligned} \max\ & f(x,y) = xy \\ \text{s.t.}\ & x + y^2 \leq 2 \\ & x, y \geq 0 \end{aligned}$$

Note that the feasible region is bounded, and $f(x,y)$ is continuous, so a global maximum exists. Rewrite the problem as

$$\begin{aligned} \min\ & -xy \\ \text{s.t.}\ & x + y^2 - 2 \leq 0 \\ & -x \leq 0 \\ & -y \leq 0 \end{aligned}$$

The KKT conditions can be written as

$$\begin{aligned} -y + \lambda_1 - \lambda_2 &= 0 \\ -x + 2\lambda_1 y - \lambda_3 &= 0 \\ x + y^2 - 2 &\leq 0 \\ -x &\leq 0 \\ -y &\leq 0 \\ \lambda_1(x + y^2 - 2) &= 0 \\ \lambda_2(-x) &= 0 \\ \lambda_3(-y) &= 0 \\ \lambda_i &\geq 0,\ i = 1, 2, 3 \end{aligned}$$

or

$$\begin{aligned}
-y + \lambda_1 - \lambda_2 &= 0 \\
-x + 2\lambda_1 y - \lambda_3 &= 0 \\
x + y^2 - 2 &\leq 0 \\
\lambda_1(x + y^2 - 2) &= 0 \\
\lambda_2 x &= 0 \\
\lambda_3 y &= 0 \\
\lambda_i &\geq 0, \ i = 1, 2, 3 \\
x, y &\geq 0
\end{aligned}$$

Suppose $\lambda_1 = 0$. Then

$$\begin{aligned}
\lambda_2 &= -y \\
\lambda_3 &= -x
\end{aligned}$$

As $x, y, \lambda_2, \lambda_3 \geq 0$, this implies $x = y = \lambda_1 = \lambda_2 = \lambda_3 = 0$.

But $f(0,0) = 0$, and it is clearly not a minimum as for instance $f(1,1) = -1$, and $(1,1)$ is feasible.

Take $\lambda_1 \neq 0$. Then, we must have $x + y^2 - 2 = 0$, and therefore x or y is strictly positive.

Suppose $x > 0$. Then $\lambda_2 = 0$ and $\lambda_1 = y$. Since $\lambda_1 \neq 0$, $\lambda_3 = 0$, and $x = 2\lambda_1 y = 2y^2$. Thus

$$0 = x + y^2 - 2 = x + \frac{x}{2} - 2 = \frac{3x}{2} - 2$$

and

$$x = \frac{4}{3}, \ y = \sqrt{\frac{2}{3}}$$

Suppose $x = 0$, $y > 0$. Thus, $y = \sqrt{2}$ and $\lambda_3 = 0$. But this also implies $\lambda_1 = 0$, while we have assumed $\lambda_1 \neq 0$. Therefore, this case cannot happen.

Therefore, we have two KKT points: $\left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right)$ and $(0,0)$. $\left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right)$ is the minimizer of the function.

Can we verify it using second-order optimality conditions? First, express $\nabla^2_{xx} L(x, \lambda)$. We have

$$\nabla^2_{xx} L(x, \lambda) = \begin{pmatrix} 0 & -1 \\ -1 & 2\lambda_1 \end{pmatrix}$$

Since the first principal minor is 0, the matrix cannot be positive definite.

For $(0,0)$, we have two active constraints:

$$\begin{aligned}
-x &= 0 \\
-y &= 0
\end{aligned}$$

The Jacobian associated to these constraints is

$$J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the LICQ is obviously verified. We can also check it by computing the rank of J:

[1]:
```julia
using LinearAlgebra

J = [-1 0; 0 -1]
rank(J)
```

[1]: 2

[3]:
```julia
J = [-1 1+1e-12; 1 -1]
rank(J,1e-8), rank(J)
```

[3]: (1, 2)

[6]:
```julia
J = [-1 1+1e-12; 1-1e-8 -1]
rank(J,1e-8), rank(J)
```

[6]: (1, 2)

[7]:
```julia
eigen(J)
```

[7]:
```
Eigen{Float64,Float64,Array{Float64,2},Array{Float64,1}}
eigenvalues:
2-element Array{Float64,1}:
 -1.9999999950005
 -4.9995000361846564e-9
eigenvectors:
2×2 Array{Float64,2}:
 -0.707107  0.707107
  0.707107  0.707107
```

[8]:
```julia
methods(rank)
```

[8]:
```
# 6 methods for generic function "rank":
[1] rank(S::SparseArrays.SparseMatrixCSC) in SuiteSparse.SPQR at C:\cygwin\home\
Administrator\buildbot\worker\package_win64\build\usr\share\julia\stdlib\v1.2\Su
iteSparse\src\spqr.jl:349
[2] rank(A::AbstractArray{T,2} where T; atol, rtol) in LinearAlgebra at C:\cygwi
n\home\Administrator\buildbot\worker\package_win64\build\usr\share\julia\stdlib\
v1.2\LinearAlgebra\src\generic.jl:838
[3] rank(x::Number) in LinearAlgebra at C:\cygwin\home\Administrator\buildbot\wo
rker\package_win64\build\usr\share\julia\stdlib\v1.2\LinearAlgebra\src\generic.j
l:843
[4] rank(C::CholeskyPivoted) in LinearAlgebra at C:\cygwin\home\Administrator\bu
ildbot\worker\package_win64\build\usr\share\julia\stdlib\v1.2\LinearAlgebra\src\
cholesky.jl:511
[5] rank(A::AbstractArray{T,2} where T, tol::Real) in LinearAlgebra at C:\cygwin
\home\Administrator\buildbot\worker\package_win64\build\usr\share\julia\stdlib\v
1.2\LinearAlgebra\src\deprecated.jl:4
```

```
[6] rank(F::SuiteSparse.SPQR.QRSparse) in SuiteSparse.SPQR at C:\cygwin\home\Administrator\buildbot\worker\package_win64\build\usr\share\julia\stdlib\v1.2\SuiteSparse\src\spqr.jl:348
```

The second-order conditions involve the computation of $d^T \nabla^2_{xx} L(x^*, \lambda^*) d$ for all $d \in N^+$, where

$$N^+ = \left\{ d \neq 0 \;\middle|\; \begin{array}{ll} d^T \nabla g_i(x^*) = 0, & i \in \mathcal{E} \\ d^T \nabla g_i(x^*) \leq 0, & i \in \mathcal{A}(x^*) \cap \mathcal{I} \end{array} \right\}$$

Unfortunately as we have also $\lambda^* = 0$ while $\mathcal{A}(x^*) \neq \emptyset$, the strict complementarity condition does not hold. It is then not trivial to characterize N^+.

It is nevertheless easy to find a $d \in N^+$ such that the second-order conditions are violated.

Note that the Jacobian matrix is

$$J(x) = \left(\nabla^T g_i(x^*), \text{ for } i \in \mathcal{A}(x^*) \right)$$

Take indeed $d = (1, 1)$. Then Jd gives

[9]:
```
d = [1.0; 1.0]
J = [-1.0 0; 0 -1.0]
J*d
```

[9]:
```
2-element Array{Float64,1}:
 -1.0
 -1.0
```

If we compute $d^T \nabla^2_{xx} L(x, \lambda) d$, we obtain

[10]:
```
D2L = [0 -1.0; -1.0 0]
d'*D2L*d
```

[10]:
```
-2.0
```

In others terms, (0,0) is not a second-order critical solution.

The Lagrange multipliers associated to $\left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right)$ is

$$\lambda^* = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \end{pmatrix}$$

and the active constraint is

$$x + y^2 - 2 = 0$$

The Jacobian of the active set at $\left(\frac{4}{3}, \sqrt{\frac{2}{3}}\right)$ is

$$J = \begin{pmatrix} 1 & 2\sqrt{\frac{2}{3}} \end{pmatrix}$$

and again, it is trivial to verify the LICQ.

But now,

$$\nabla^2_{xx} L(x^*, \lambda^*) = \begin{pmatrix} 0 & -1 \\ -1 & 2\sqrt{\frac{2}{3}} \end{pmatrix}$$

and the strict complementarity condition holds.

Thus,

$$N^+ = \{d \neq 0 \,|\, Jd = 0\}\,.$$

Therefore, we have to consider the vectors $d \in \mathbb{R}^n$ such that

$$d^T \begin{pmatrix} 1 \\ 2\sqrt{\frac{2}{3}} \end{pmatrix} = 0$$

In other words, $d \in Null(J)$, $d \neq 0$, where

$$J = \begin{pmatrix} 1 & 2\sqrt{\frac{2}{3}} \end{pmatrix}$$

[6]:
```
A = [1 2*sqrt(2/3) ]
w = nullspace(A)
```

[6]:
```
2×1 Array{Float64,2}:
 -0.8528028654224418
  0.5222329678670935
```

w is a basis vector of A, of norm equal to 1:

[7]:
```
norm(w)
```

[7]:
```
1.0
```

But

[8]:
```
D2L[2,2] = 2*sqrt(2/3)
w'*D2L*w
```

[8]:
```
1×1 Array{Float64,2}:
 1.33608531424537
```

Let $d = \sum_i {}_i w_i$, $d \neq 0$. Then

$$d^T \nabla^2_{xx} L(x, \lambda) d = \sum_i {}_i^2 w_i^T \nabla^2_{xx} L(x, \lambda) w_i > 0.$$

The necessary and sufficient second-order optimality conditions are then satisfied.

1.3 Example 2

Use the Karush-Kuhn-Tucker conditions to solve

$$\begin{aligned} &\max KL \\ &\text{subject to } 4K + L \leq 8 \\ &K, L \geq 0 \end{aligned}$$

The KKT conditions are

$$\begin{aligned} L - 4\lambda_1 + \lambda_2 &= 0 \\ K - \lambda_1 + \lambda_3 &= 0 \\ \lambda_1(8 - 4K - L) &= 0 \\ \lambda_2 K &= 0 \\ \lambda_3 L &= 0 \\ 4K + L &\leq 8 \\ K, L, \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

1.3.1 Case 1.

If $\lambda_1 = 0$, the first KKT condition says $L + \lambda_2 = 0$, which implies $L = \lambda_2 = 0$, and the second says $K + \lambda_3 = 0$, which implies $K = \lambda_3 = 0$. The KKT conditions are indeed satisfied with $K = L = \lambda_1 = \lambda_2 = \lambda_3 = 0$, and the objective value at K = L = 0 is 0.

1.3.2 Case 2

If $\lambda_1 > 0$, $4K + L = 8$. Thus at least one of K and L is positive, implying that λ_2 or λ_3 is 0. If $\lambda_2 = 0$, $L = 4\lambda_1 > 0$, but that implies $\lambda_3 = 0$. Similarly, if $\lambda_3 = 0$, $K = \lambda_1 > 0$, but that implies $\lambda_2 = 0$. So we must have $\lambda_2 = \lambda_3 = 0$, $L = 4\lambda_1$ and $K = \lambda_1$. Then $4K + L = 8$, $K = \lambda_1$, $L = \lambda_1$, implying $4\lambda_1 + 4\lambda_1 = 8$, so $\lambda_1 = 1$, $K = 1$ and $L = 4$. The KKT conditions are satisfied with $K = 1$, $L = 4$, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 0$, and the objective value is 4.

[ ]: